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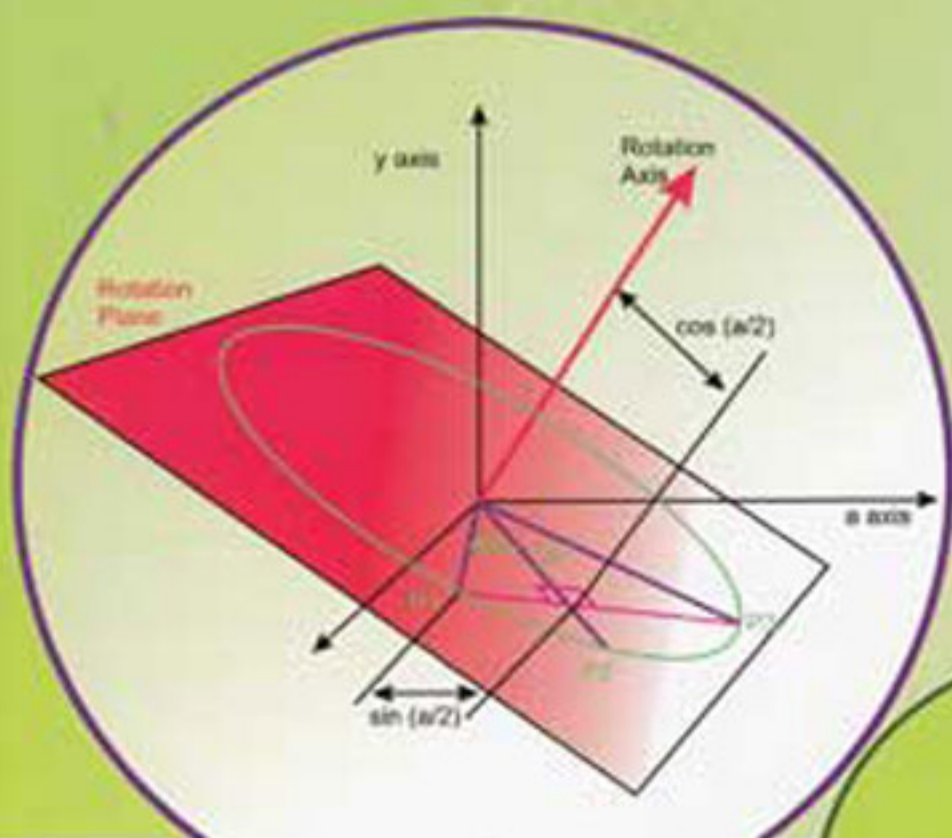
**MATHEMATICAL  
METHODS IN PHYSICS**

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S. D. AGHAV            Dr. B. G. WAGH

S. Y. B. Sc. PHYSICS • PAPER I - SEMESTER I



**A Book Of**

# **MATHEMATICAL METHODS IN PHYSICS**

**S. Y. B. Sc. Physics : Semester-I : Paper-I (PH 211)**

**As Per Revised Syllabus Effective from June 2014**

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## **Preface ...**

This book entitled "**Mathematical Methods in Physics**" has been written for the courses in Physics at first semester of S.Y.B.Sc. according to new syllabus to be implemented from June 2014.

The text book is designed for the students preparing for S.Y.B.Sc. examination of Savitribai Phule Pune University. The purpose of this text book is to give systematic exposition of revised syllabus covering all the topics based on UGC model of curriculum.

Beginning with its fundamental, the subject has been developed systematically and logically with the emphasis on the physical explanation supported adequately by mathematical formulation. The clarity and systematic representation will make the book intelligible to the beginner.

Numerical examples have been added at the end of topic to illustrate applications of theoretical principles.

Authors have done sincere efforts to explain all topics in syllabus in a simple and lucid language as possible. Although the book is not voluminous, but everything useful and important information has been added. Authors sincerely feel that the work will adequately meet the needs of S.Y.B.Sc. students for a good and concise book which will stimulate a genuine interest among students in the subject.

Authors sincerely thank Shri. Dineshbhai Furia, Mr. Jignesh Furia, Shri. M. P. Munde and entire staff of Nirali Prakashan especially Mr. Santosh Bare, Mr. Kiran Velankar and Mrs. Prachi Sawant for their co-operation in completion of the book in very short span of time.

Sufficient care has been taken to avoid misprints, checking solutions and answers of numerical examples. However, authors will most gratefully accept suggestions for improving the book and making it more useful for students and teachers.

**JUNE 2014**

**AUTHORS**

**PUNE**

# Syllabus ...

## **Semester-I (P-I) : Mathematical Methods in Physics**

### **1. Complex Numbers (12)**

- 1.1 Introduction to complex numbers.
- 1.2 Rectangular, polar and exponential forms of complex numbers.
- 1.3 Argand diagram
- 1.4 Algebra of complex numbers using mathematical and Argand diagram
- 1.5 De-Moivre's theorem
- 1.6 Powers, roots and log of complex numbers.
- 1.7 Trigonometric, hyperbolic and exponential functions.
- 1.8 Applications of complex numbers to determine velocity and acceleration in curved motion
- 1.9 Problems

### **2. Partial Differentiation (12)**

- 2.1 Definition of partial differentiation
- 2.2 Successive differentiation
- 2.3 Total differentiation
- 2.4 Exact differential
- 2.5 Chain rule
- 2.6 Theorems of differentiation
- 2.7 Change of variables from Cartesian to polar co-ordinates.
- 2.8 Implicit and explicit functions
- 2.9 Conditions for maxima and minima (without proof)
- 2.10 Problems.

### **3. Vector Algebra (06)**

- 3.1 Introduction to scalars and vectors
- 3.2 Dot product and cross product of two vectors and its physical significance
- 3.3 Scalar triple product and its geometrical interpretation.
- 3.4 Vector triple product and its proof.
- 3.5 Problems.

### **4. Vector Analysis (12)**

- 4.1 Introduction
- 4.2 Scalar and vector fields
- 4.3 Differentiation of vectors with respect to scalar
- 4.4 Vector differential operator and Laplacian operator
- 4.5 Gradient of scalar field and its physical significance.
- 4.6 Divergence of scalar field and its physical significance

4.7 Curl of vector field

4.8 Vector identities

(a)  $\nabla \times \nabla\phi = 0$

(b)  $\nabla \cdot (\nabla \times V) = 0$

(c)  $\nabla \cdot (\nabla\phi) = \nabla^2\phi$

(d)  $\nabla \cdot (\phi A) = \nabla\phi \cdot A + \phi (\nabla \cdot A)$

(e)  $\nabla \times (\phi A) = \phi(\nabla \times A) + (\nabla\phi) \times A$

(f)  $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

4.9 Problems.

## 5. Differential Equations

(06)

5.1 Frequently occurring partial differential equations (Cartesian coordinates)

5.2 Degree, order, linearity and homogeneity of differential equation.

5.3 Concept of singular points. Examples of singular points ( $x = 0$ ,  $x = x_0$  and  $x = \infty$ ) of differential equations.

5.4 Problems.

### Additional Activity :

Four tutorials containing 10 unsolved problems each from suggested references.

...

### Reference Books :

1. Methods of Mathematical Physics by Laud, Takwale and Gambhir
2. Mathematical Physics by B. D. Gupta
3. Mathematical Physics by Rajput and Gupta
4. Mathematical Methods in Physical Science by Mary and Boas
5. Vector analysis by Spiegel and Murrey
6. Mathematical Methods for Physicists by Arfken and Weber, 5<sup>th</sup> Edition, Academic Press.

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# Chapter 1...

## Complex Numbers

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- 1.1 Introduction to complex numbers
  - 1.2 Rectangular, polar and exponential forms of complex numbers
  - 1.3 Argand diagram
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  - 1.5 De Moivre's theorem (statement only)
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### 1.1 Introduction to Complex Numbers

- In algebra, we are familiar with a quadratic equation as given below

$$az^2 + bz + c = 0 \text{ with } a \neq 0 \quad \dots (1.1)$$

where  $a$ ,  $b$  and  $c$  are constants.

- The roots of the above equation i.e. unknown values of  $z$  are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \dots (1.2)$$

The term  $b^2 - 4ac$  in the above equation is discriminant.

- Consider the quadratic equation

$$4z^2 - 20z + 15 = 0 \quad \dots (1.3)$$

The discriminant  $d = b^2 - 4ac = 160$  is positive and the roots of the above equation are

$$\frac{20 \pm \sqrt{160}}{8} = \frac{5 \pm \sqrt{10}}{2} \quad \dots(1.4)$$

Thus, the roots are  $\frac{5 + \sqrt{10}}{2}$  and  $\frac{5 - \sqrt{10}}{2}$ .

These roots are real irrational numbers.

- Let us consider another quadratic equation

$$4z^2 - 20z + 27 = 0 \quad \dots (1.5)$$

The discriminant  $d = b^2 - 4ac = -32$ , which is negative.

The roots of the above equation are given by

$$\frac{20 \pm \sqrt{-32}}{8} = \frac{5 \pm \sqrt{-2}}{2} \quad \dots (1.6)$$

(1.1)

- The modulus of a complex number is also known as an absolute value.

e.g. If

$$z = 3 - 2i \text{ then}$$

$$|z| = \sqrt{(3)^2 + (-2)^2} = \sqrt{13}$$

### 1.1.3 Equality of Two Complex Numbers

(April 16)

- Two complex numbers are said to be equal if and only if their real and imaginary parts are separately equal.
- The complex numbers  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  are said to be equal if and only if  $x_1 = x_2$  and  $y_1 = y_2$ .
- Complex numbers have great importance because these numbers are not only useful in Mathematics but also in Physics and Engineering Sciences. In Physics, approach with use of complex number gives an easy solution or interpretation involved in the situation. e.g. analytical treatment of A.C. circuits, study of forced oscillations of a mechanical (dynamical) system, intensity of light due to interference of multiple reflections in a thin film or of transmitted beam of light through plane diffraction grating, probability calculations in Quantum Mechanics and Statistical Mechanics, etc.
- In some books to avoid confusion between the notation  $i$  for electric current and  $i$  for imaginary number, the notation  $j$  is used instead of  $i$  in complex number. Hence with this notation, complex number  $z$  is expressed as

$$z = x + jy \quad \dots (1.11)$$

- In the further course of this chapter, we will understand mathematical (algebraic) operations in case of complex numbers, their graphical representation (Argand diagram), some trigonometrical, exponential complex functions.

### Algebra of Complex Numbers

- If  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  are two complex numbers, then various mathematical operations are carried out in the following ways :

**(i) Addition :**

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

**(ii) Subtraction :**

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

If

$$z_1 = z_2 = z \text{ then}$$

$$\begin{aligned} z_1 - z_2 &= z - z \\ &= (x + iy) - (x + iy) \\ &= 0 + i0 \end{aligned}$$

Thus the complex number is equal to zero when  $x = 0$  and  $y = 0$ .

(iv) Associativity of multiplication :

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(v) Distributivity of multiplication :

$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$$

The real numbers also have the above mentioned properties.

(vi) Equality of complex numbers :

If  $z_1$  and  $z_2$  are equal then

$$x_1 + iy_1 = x_2 + iy_2$$

$$\therefore (x_1 - x_2) + i(y_1 - y_2) = 0 \quad \text{as } z_1 = z_2.$$

Hence, the complex number on L.H.S. will be zero when simultaneously  $(x_1 - x_2) = 0$  and  $(y_1 - y_2) = 0$ .

$$\therefore x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

Hence two complex numbers are equal when their real parts and imaginary parts are equal.

## 1.2 Rectangular, Polar and Exponential Forms of Complex Numbers

(April 13, Oct. 14)

**(i) Rectangular form of a complex number :** In the form  $x + iy$  of a complex number, it is expressed by  $x$  and  $y$  rectangular co-ordinates, hence it is known as the *rectangular form* of a complex number. **(Oct. 14)**

The Argand diagram shows that

$$OP = \sqrt{x^2 + y^2} = |z| \quad \dots (1.12)$$

The point  $P(x, y)$  on  $XY$  plane represents one and only one complex number  $z$ , so  $OP$  can be regarded as position vector of point  $P$  and shown as vector  $\vec{OP}$ .

The points on  $X$  axis have only real part and those on  $Y$  axis have only imaginary part.

**(ii) Polar form of a complex number :** In two-dimensional analytical geometry, a system of Cartesian co-ordinates  $(x, y)$  or polar co-ordinates  $(r, \theta)$  are used. In Fig. 1.1,  $OX$  and  $OY$  are rectangular co-ordinate axes with  $O$  as the origin. Consider a point  $P(x, y)$  at a distance  $r = OP'$ . Let  $\theta$  be the angle made by  $r$  with  $X$  axis. Therefore

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\text{and} \quad x^2 + y^2 = r^2$$

Using the polar co-ordinates  $(r, \theta)$ , a complex number  $z$  corresponding to  $P'$  is given by

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \end{aligned}$$

$$\therefore z = r (\cos \theta + i \sin \theta) \quad \dots(1.13)$$

$$\begin{aligned} \text{Also} \quad \cos \theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \\ &= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \dots \end{aligned} \quad \text{where } i^2 = -1 \text{ and } i^4 = 1$$

$$\text{and} \quad \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\therefore \quad i \sin \theta = i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \dots$$

$$\therefore \quad \cos \theta + i \sin \theta = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$$\therefore \quad \cos \theta + i \sin \theta = e^{i\theta}$$

This is known as Euler's formula. So the complex number becomes

$$z = r (\cos \theta + i \sin \theta) = r e^{i\theta} \quad \dots (1.16)$$

This form of  $z$  is called the exponential form. As before,  $r$  is the modulus and  $\theta$  is the argument. In this form,  $z$  is expressed as the product of the real part  $r$  and the imaginary part  $e^{i\theta}$ .

### 1.3 Argand Diagram

(April 16, Oct. 14)

- Even though a complex number is essentially an algebraic quantity, it can be conveniently represented graphically i.e. geometrically. Fig. 1.2 shows a system of two rectangular axes  $X$  and  $Y$ . A complex number consists of (i) real part and (ii) imaginary part. Real part is plotted along  $X$  axis, hence it is known as the axis of reals. Imaginary part is plotted along  $Y$  axis, hence it is known as axis of imaginaries. Any point on  $X$  axis has real value  $x$  and that on  $Y$  axis has imaginary value  $iy$ , where  $y$  is real but  $iy$  is pure imaginary.

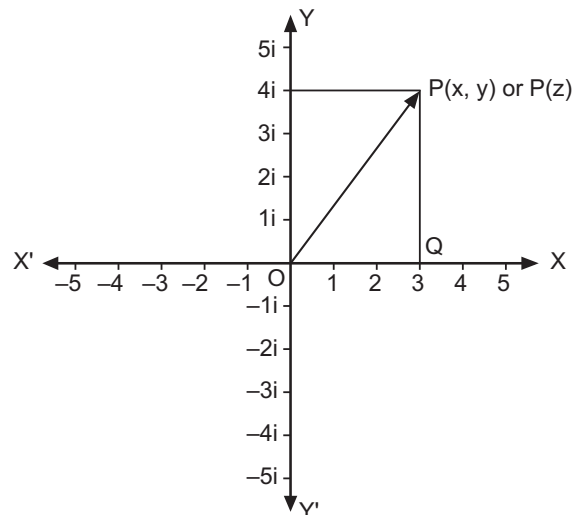


Fig. 1.2 : Argand diagram

Thus the complex number  $\left(\frac{z_1}{z_2}\right)$  is represented by the point R which has polar co-ordinates  $\left(\frac{r_1}{r_2}\right)$  and  $(\theta_1 - \theta_2)$ . The point R is obtained such that

$$OR = \frac{OP}{OQ} = \frac{r_1}{r_2}$$

and it makes an angle equal to  $(\theta_1 - \theta_2)$  with the real axis.

### 1.5 De Moivre's Theorem (Statement only)

(April 15, 12)

This theorem is useful to find the powers and roots of a complex number.

**Statement :**

$$(\cos \theta + i \sin \theta)^n = \cos (n\theta) + i \sin (n\theta)$$

where n is a power.

Now  $(\cos \theta + i \sin \theta) = e^{i\theta}$

$$\begin{aligned} \therefore (\cos \theta + i \sin \theta)^n &= (e^{i\theta}) (e^{i\theta}) (e^{i\theta}) \dots n \text{ times} \\ &= e^{i\theta + i\theta + i\theta + \dots + n \text{ terms}} \\ &= e^{in\theta} \\ &= e^{i(n\theta)} \end{aligned}$$

$$\therefore (\cos \theta + i \sin \theta)^n = \cos (n\theta) + i \sin (n\theta)$$

Above equation represents De Moivre's theorem.

### 1.6 Power, Roots and Logarithm of Complex Numbers

#### 1.6.1 Power of a Complex Number

Consider a complex number  $z = x + iy$ . The  $n^{\text{th}}$  power of a complex number  $z$  is  $z^n$ .

In exponential form, the complex number  $z = x + iy$  can be expressed as

$$z = r e^{i\theta}$$

$$\therefore z^n = (r e^{i\theta})^n = r^n e^{in\theta} \quad \dots (1.22)$$

Using the expression of De Moivre's theorem, equation (1.22) can be written as

$$z^n = r^n (\cos n\theta + i \sin n\theta) \quad \dots (1.23)$$

Equation (1.23) gives

$$|z^n| = r^n \text{ i.e. } |z^n| = |z|^n$$

Also  $\text{Arg}(z^n) = n \text{Arg} z$

The powers of complex number are multiplied values because

$$e^{i\theta} = e^{i(\theta + 2n\pi)} \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Hence, while determining powers of complex numbers, powers are restricted to only principle values.

In above equation,  $\theta$  has an infinite number of values (all differing by multiple of  $2\pi$ ), so a complex number  $p = \ln z$  has infinitely many logarithms differing from each other by a multiple of  $2\pi$ .

## 1.7 Trigonometric, Exponential and Hyperbolic Functions

### 1.7.1 Trigonometric Functions

(April 2013)

Using Euler's formula from Article 1.2, we have

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \dots (1.24)$$

Replacing  $\theta$  by  $-\theta$  in above equation, we get

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta)$$

But,  $\cos(-\theta) = \cos \theta$  and  $\sin(-\theta) = -\sin \theta$

$$\therefore e^{-i\theta} = \cos \theta - i \sin \theta \quad \dots (1.25)$$

Adding equations (1.24) and (1.25), we get

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \dots (1.26)$$

By subtracting equation (1.25) from equation (1.24), we get

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \dots (1.27)$$

Using above equations, we can obtain following relations :

$$\begin{aligned} \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} & \sec \theta &= \frac{2}{e^{i\theta} + e^{-i\theta}} \\ \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} & \operatorname{cosec} \theta &= \frac{2i}{e^{i\theta} - e^{-i\theta}} \\ \tan \theta &= \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})} & \cot \theta &= \frac{i(e^{i\theta} + e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} \end{aligned}$$

Above equations give formulae for trigonometric functions of real angles.

When real  $\theta$  is replaced by a complex variable  $z$ , then Euler's formula gives

$$e^{iz} = \cos z + i \sin z$$

$$e^{-iz} = \cos z - i \sin z$$

With complex variable  $z$ , the relations of sine, cosine, etc. are similar as that for real variable  $\theta$  i.e.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Similarly, other relations of tangent, etc. can be obtained.

**1.7.2 Hyperbolic Functions****(Oct. 14)**

We have shown that  $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$  ... (1.28)

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \dots (1.29)$$

If the complex number  $z$  is pure imaginary with its real part zero then  $z = iy$ .

Using  $z = iy$ , equations (1.28) and (1.29) may be written as

$$\sin(iy) = \frac{e^{-y} - e^y}{2i}$$

Or  $\sin(iy) = \frac{i(e^y - e^{-y})}{2}$  ... (1.30)

Similarly,  $\cos(iy) = \frac{e^{-y} + e^y}{2}$

Or  $\cos iy = \frac{e^y + e^{-y}}{2}$  ... (1.31)

The above functions are called hyperbolic functions and are abbreviated as  $\sinh$ ,  $\cosh$  etc. The definitions for  $z$  are :

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \dots (1.32)$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \dots (1.33)$$

The other hyperbolic functions such as  $\tanh z$ ,  $\operatorname{sech} z$ , etc. are defined in similar way parallel to trigonometric functions i.e.

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \operatorname{coth} z = \frac{\cosh z}{\sinh z}$$

$$\operatorname{cosech} z = \frac{1}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}$$

**1.7.3 Some Important Identities****(April 15, Oct. 15)**

(i)  $\sinh(i\theta) = i \sin \theta$

Using  $z = i\theta$  in equation (1.32), we get

$$\sinh(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} = i \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)$$

$\therefore \sinh(i\theta) = i \sin \theta$

(ii)  $\cosh(i\theta) = \cos \theta$

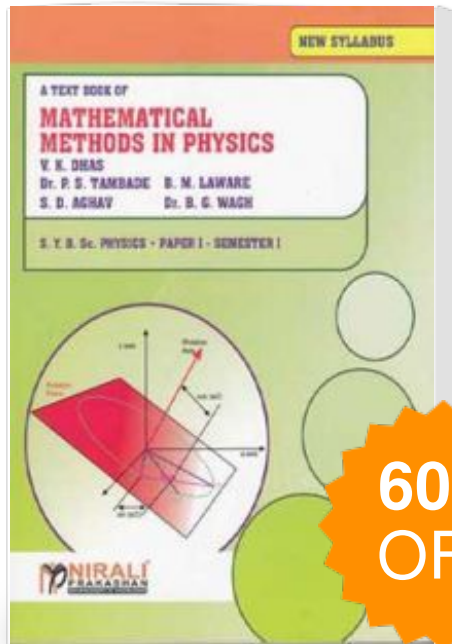
**(Oct. 14)**

Using  $z = i\theta$  in equation (1.33), we get

$$\cosh(i\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$\therefore \cosh(i\theta) = \cos \theta$

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