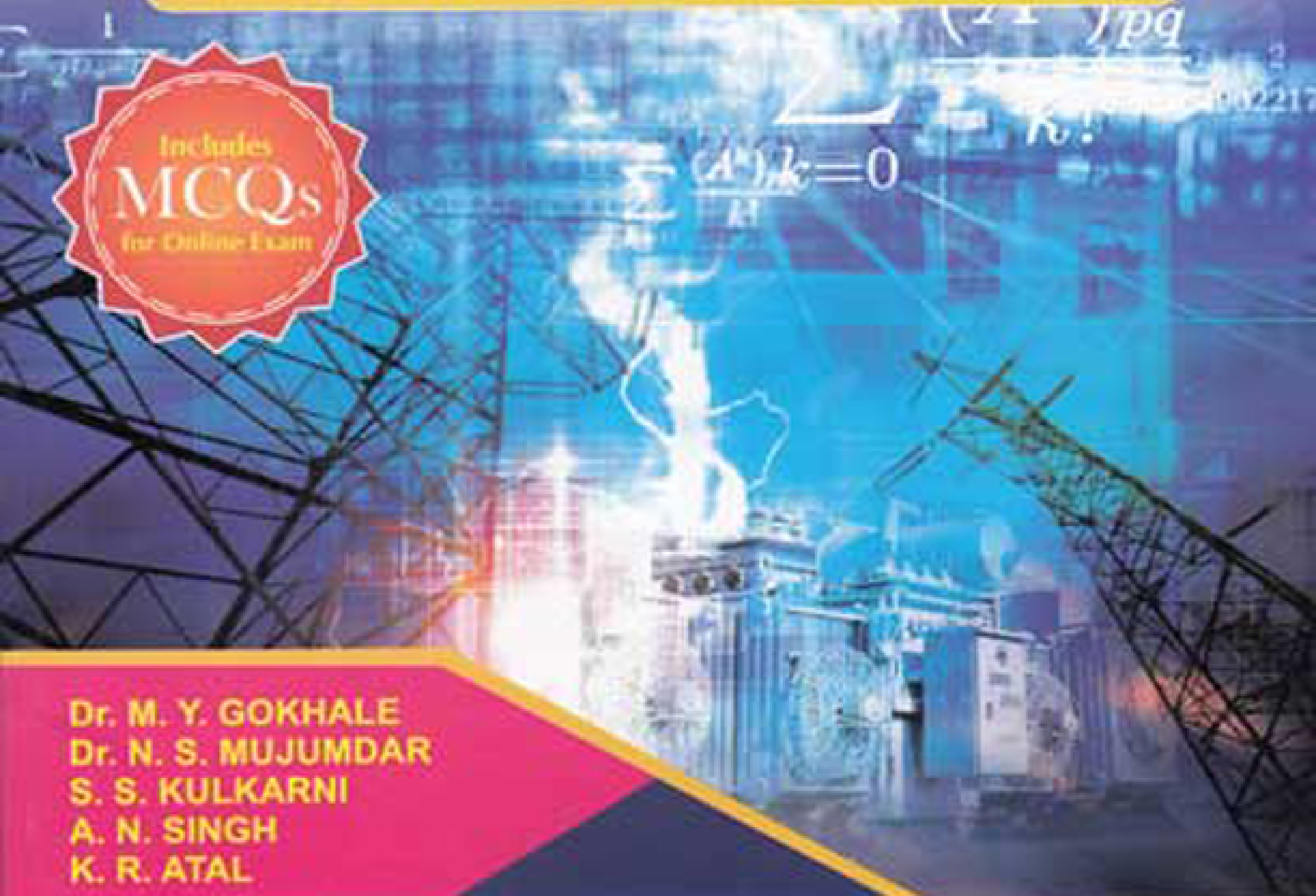
SPPU According to New Revised Credit System Syllabus

Second Year Degree Course In
ELECTRICAL, ELECTRICAL SANDWITCH &
INSTRUMENTATION AND CONTROL ENGG. (Sem - I)

ENGINEERING MATHEMATICS - III

With Large No. Of Solved Problems & Univ Question Papers



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ENGINEERING MATHEMATICS – III

(With Large Number of MCQ's & Solved Problems)

FOR

SEMESTER - I

S.E. : SECOND YEAR DEGREE COURSE IN ELECTRICAL, ELECTRICAL SANDWITCH & INSTRUMENTATION AND CONTROL ENGINEERING

ACCORDING TO NEW REVISED CREDIT SYSTEM SYLLABUS OF SAVITRIBAI PHULE PUNE UNIVERSITY

(EFFECTIVE FROM ACADEMIC YEAR – JUNE 2016)

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PREFACE TO THE SECOND EDITION

We are glad and excited to announce that the First Edition of this book received an overwhelming response from the engineering student community, compelling us to release its **Second Edition** within a very short period of time.

This thoroughly revised **Second Edition** has been updated with additional matter, many solved problems, including solutions to all University Examination Problems and Numerous Exercises for practice.

Special care has been taken to maintain high degree of accuracy in the theory and numericals throughout the book.

We take this opportunity to express our sincere thanks to Dineshbhai Furia of Nirali Prakashan, pioneer in all fields of education. Our special thanks to Jignesh Furia for their effective cooperation and great care in bringing out this revised edition. We also appreciate the efforts of M. P. Munde and the entire staff of Engineering Books Deptt. of Nirali Prakashan namely Mrs. Deepali Lachake (Co-ordinator) for bringing this book to the students in a timely manner.

We sincerely hope that this "**Second Edition**" will also be warmly received by all concerned as in the past.

Valuable suggestions from our esteemed readers to improve the book are most welcome and highly appreciated.

Pune **Authors**

PREFACE TO THE FIRST EDITION

Our text books on **Engineering Mathematics-III** have occupied place of pride among engineering student's community for more than **twenty years** now. All the teachers of this group of authors have been teaching mathematics in engineering colleges for the past several years. Difficulties of engineering students are well understood by the authors and that is reflected in the text material.

As per the policy of the University, Engineering Syllabi is revised every five years. Last revision was in the year 2012. New revision is coming little earlier, as university has introduced **online** system of examination from year 2012.

As per the new credit system, the **Insem (Online) Examinations** (Combined Phase-I and Phase-II) will be conducted based on first, second, third and fourth units. The **Online** examinations will have objective types of questions with multiple choices. End semester examination will be based on all the six units and that will be conducted in traditional way.

New text book is written, taking in to account all the new features that have been introduced. All the entrants to the engineering field will definitely find this book, complete in all respect. Students will find the subject matter presentation quite lucid. There are large number of illustrative examples and well graded exercises. **Addition of Multiple Choice Questions will be very useful to the students**, especially for **Online** examinations.

We take this opportunity to express our sincere thanks to Shri. Dineshbhai Furia of Nirali Prakashan, pioneer in all fields of education. Thanks are also due to Shri. Jignesh Furia, whose dynamic leadership is helpful to all the authors of Nirali Prakashan.

We have no doubt that like our earlier texts, student's community will respond favourably to this new venture.

The advice and suggestions of our esteemed readers to improve the text are most welcomed, and will be highly appreciated.

15th June 2016 **Authors**

Pune

SYLLABUS

Unit I : Linear Differential Equations (LDE) and Applications : (09 Hrs.)

LED of n^{th} order with constant coefficients, Method of variation of parameters, Cauchy's and Legendre's DE, Simultaneous and Symmetric simultaneous DE. Modeling of electrical circuit.

Unit II : Laplace Transforms (09 Hrs.)

Definition of LT, Inverse LT, Properties and theorems, LT of standard functions, LT of some special function viz. Periodic, Unit step, Unit impulse. Applications of LT for solving linear differential equations.

Unit III : Fourier and Z-transforms (09 Hrs.)

Fourier Transform (FT) : Complex exponential form of Fourier series, Fourier integral theorem, Sine and Cosine integrals, Fourier transform, Fourier sine and Cosine transform and their inverses.

Z-transform (ZT) : Introduction, Definition, Standard properties, ZT of standard sequences and their inverses. Solution of difference equations.

Unit IV : Vector Differential Calculus (09 Hrs.)

Physical interpretation of Vector differentiation, Vector differential operator, Gradient, Divergence and Curl, Directional derivative, Solenoidal, Irrotational and Conservative fields, Scalar potential, Vector identities.

Unit V : Vector Integral Calculus and Applications (09 Hrs.)

Line, Surface and Volume integrals, Work-done, Green's Lemma, Gauss's Divergence theorem, Stoke's theorem. Applications to problems in Electro-magnetic fields..

Unit VI : Complex Variables (09 Hrs.)

Functions of complex variables, Analytic functions, Cauchy-Riemann equations, Conformal mapping, Bilinear transformation, Cauchy's integral theorem, Cauchy's integral formula, Laurent's series, Residue theorem.

CONTENTS

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UNIT - I : LINEAR DIFFERENTIAL EQUATIONS AND APPLICATIONS

CHAPTER ONE

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1.1 INTRODUCTION

Differential equations are widely used in fields of Engineering and Applied Sciences. Mathematical formulations of most of the physical problems are in the forms of differential equations. Use of differential equations is most prominent in subjects like Circuit Analysis, Theory of Structures, Vibrations, Heat Transfer, Fluid Mechanics etc. Differential equations are of two types : Ordinary and Partial Differential Equations. In ordinary equations, there is one dependent variable depending for its value on one independent variable. Partial differential equations will have more than one independent variables.

In what follows, we shall discuss ordinary and partial differential equations, which are of common occurrence in engineering fields. Applications to some areas will also be dealt.

1.2 PRELIMINARIES

I. Second Degree Polynomials and Their Factorization :

(a)

(i) $D^2 - 2D - 3 = (D + 1)(D - 3)$ (ii) $D^2 + 5D + 6 = (D + 2)(D + 3)$

(iii) $D^2 + 2D + 1 = (D + 1)^2$ (iv) $D^2 - 5D + 6 = (D - 2)(D - 3)$

(v) $D^2 + 3D + 2 = (D + 2)(D + 1)$ (vi) $D^2 - D - 2 = (D - 2)(D + 1)$

(vii) $D^2 - 4D + 4 = (D - 2)^2$ (viii) $D^2 - a^2 = (D - a)(D + a)$

(ix) $D^2 + a^2 = (D + ia)(D - ia)$

(b) The roots of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, these roots are imaginary if $b^2 - 4ac < 0$.

(i) $D^2 + 2D + 2 = 0 \Rightarrow D = \dfrac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$

(ii) $D^2 + D + 1 = 0 \Rightarrow D = \dfrac{-1 \pm \sqrt{1 - 4}}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{3}}{2} i$

If $D = \dfrac{-1}{2} \pm i \dfrac{\sqrt{3}}{2} = \alpha \pm i\beta$ then $\alpha = -\dfrac{1}{2}$, $\beta = \dfrac{\sqrt{3}}{2}$, β is always positive; α may be positive, negative or zero.

(iii) $D^2 + 1 = 0 \Rightarrow D^2 = -1$ i.e. $D = \pm i$ $\therefore$ $\alpha = 0, \beta = 1$.

(iv) $D^2 + 4 = 0 \Rightarrow D^2 = -4$ i.e. $D = \pm 2i$ $\therefore$ $\alpha = 0, \beta = 2$.

II. Third Degree Polynomials and Their Factorization :

(a) (i) $D^3 - a^3 = (D - a)(D^2 + aD + a^2)$

(ii) $D^3 + 3D^2 + 3D + 1 = (D + 1)^3$

(iii) $D^3 + a^3 = (D + a)(D^2 - aD + a^2)$

(iv) $D^3 - 3D^2 + 3D - 1 = (D - 1)^3$

(b) Use of synthetic division :

(i) $f(D) = D^3 - 7D - 6 = 0$; for $D = -1$, $f(-1) = 0$ $\therefore$ $(D + 1)$ is one of the factors.

– 1	1	0	– 7	– 6
		– 1	1	6
	1	– 1	– 6	0

$\therefore$ $D^3 - 7D - 6 = 0 \Rightarrow (D + 1)(D^2 - D - 6) = 0$

$(D + 1)(D - 3)(D + 2) = 0 \Rightarrow D = -1, -2, 3.$

(ii) For $D^3 - 2D + 4 = 0$; $D = -2$ $\therefore$ $f(-2) = 0$ $\therefore$ $(D + 2)$ is one of the factors.

– 2	1	0	– 2	4
		– 2	4	– 4
	1	– 2	2	0

$\therefore$ $D^3 - 2D + 4 = 0 \Rightarrow (D + 2)(D^2 - 2D + 2) = 0$

$D = -2$ and $D = 1 \pm i$, $\alpha = 1$, $\beta = 1$.

III. Fourth Degree Polynomials and Their Factorization :

(a) $D^4 - a^4 = (D^2 - a^2)(D^2 + a^2) = (D - a)(D + a)(D + ia)(D - ia)$

(b) Making a perfect square by introducing a middle term :

(i) For $D^4 + a^4 = 0$; consider $(D^2 + a^2)^2 = D^4 + 2a^2 D^2 + a^4$

$$D^4 + a^4 = (D^4 + 2a^2 D^2 + a^4) - (2a^2 D^2) = (D^2 + a^2)^2 - \left(\sqrt{2}\, a D\right)^2$$

$$D^4 + a^4 = \left(D^2 - \sqrt{2}\, a D + a^2\right)\left(D^2 + \sqrt{2}\, a D + a^2\right)$$

(ii) For $D^4 + 1 = D^4 + 2D^2 + 1 - 2D^2 = (D^2 + 1)^2 - \left(\sqrt{2}\, D\right)^2$

$$D^4 + 1 = \left(D^2 - \sqrt{2}\, D + 1\right)\left(D^2 + \sqrt{2}\, D + 1\right)$$

(c) $D^4 + 8D^2 + 16 = (D^2 + 4)^2$, $D^4 + 2D^2 + 1 = (D^2 + 1)^2 = (D + i)^2 (D - i)^2$

$D^4 + 10D^2 + 9 = (D^2 + 9)(D^2 + 1) = (D + 3i)(D - 3i)(D + i)(D - i)$

(d) (i) $f(D) = D^4 - 2D^3 - 3D^2 + 4D + 4 = 0$, for $D = -1$, $f(-1) = 0$

– 1	1	– 2	– 3	4	4
		– 1	3	0	– 4
– 1	1	– 3	0	4	0
		– 1	4	– 4	
2	1	– 4	4	0	
		2	– 4		
	1	– 2	0		

$\therefore$ Factors are $(D + 1)^2 (D - 2)^2 = 0$.

On a similar line,

(ii) $D^4 - D^3 - 9D^2 - 11D - 4 = (D+1)^3 (D-4)$

(e) Perfect square of the type $(a + b + c)^2$

(i) $D^4 + 2D^3 + 3D^2 + 2D + 1 = (D^2)^2 + 2 \cdot D^2 \cdot D + D^2 + 2D^2 + 2D + 1$

$= (D^2 + D)^2 + 2(D^2 + D) + 1$

$= [(D^2 + D) + 1]^2 = (D^2 + D + 1)^2$

(ii) $D^4 - 4D^3 + 8D^2 - 8D + 4 = (D^2)^2 - 2D^2 \cdot 2D + (2D)^2 + 4D^2 - 8D + 4$

$= (D^2 - 2D)^2 + 4(D^2 - 2D) + 4$

$= [(D^2 - 2D) + 2]^2 = (D^2 - 2D + 2)^2$

IV. Fifth Degree Polynomials and Their Factorization :

(i) $D^5 - D^4 + 2D^3 - 2D^2 + D - 1 = D^4 (D-1) + 2D^2 (D-1) + 1 (D-1)$

$= (D^4 + 2D^2 + 1)(D-1) = (D-1)(D^2+1)^2$

$= (D-1)(D+i)^2 (D-i)^2$

1.3 THE nth ORDER LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

A differential equation which contains the differential coefficients and the dependent variable in the first degree, does not involve the product of a derivative with another derivative or with dependent variable, and in which the coefficients are constants is called a *linear differential equation with constant coefficients.*

The general form of such a differential equation of order "n" is

$$\mathbf{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = f(x)} \quad \ldots (1)$$

Here $a_0, a_1, a_2 \ldots$ are constants. Equation (1) is a nth order linear differential equation with constant coefficients.

e.g. Put n = 3 in equation (1), we get $a_0 \frac{d^3 y}{dx^3} + a_1 \frac{d^2 y}{dx^2} + a_2 \frac{dy}{dx} + a_3 y = f(x)$ which is a 3rd order linear differential equation with constant coefficients.

Using the differential operator D to stand for $\frac{d}{dx}$ i.e. $Dy = \frac{dy}{dx}$; $D^2 y = \frac{d^2 y}{dx^2}$, … $D^n y = \frac{d^n y}{dx}$, the equation (1) will take the form

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots + a_{n-1} Dy + a_n y = f(x)$$

OR $$\mathbf{(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n)\, y = f(x)} \quad \ldots (2)$$

in which each term in the parenthesis is operating on y and the results are added.

Let $\phi(D) \equiv a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n$, $\phi(D)$ is called as n^{th} order polynomial in D.

$\therefore$ Equation (2) can be written as $\boxed{\phi(D)\, y = f(x)}$ … (3)

Note : In equation (1), if $a_0, a_1, \ldots a_n$ are functions of x then it is called n^{th} order linear differential equation.

1.4 THE NATURE OF DIFFERENTIAL OPERATOR "D"

It is convenient to introduce the symbol D to represent the operation of differentiation with respect to x. i.e. $D \equiv \frac{d}{dx}$, so that

$$\frac{dy}{dx} = Dy;\ \frac{d^2 y}{dx^2} = D^2 y;\ \frac{d^3 y}{dx^3} = D^3 y;\ \ldots\ldots;\ \frac{d^n y}{dx^n} = D^n y \text{ and } \frac{dy}{dx} + ay = (D + a)\, y$$

The differential operator D or (D^n) obeys the laws of Algebra.

Properties of the operator D :

If y_1 and y_2 are differentiable functions of x and "a" is a constant and m, n are positive integer then

(i) $D^m (D^n)\, y = D^n (D^m)\, y = D^{m+n}\, y$

(ii) $(D - m_1)(D - m_2)\, y = (D - m_2)(D - m_1)\, y$

(iii) $(D - m_1)(D - m_2)\, y = [D^2 - (m_1 + m_2) D + m_1 m_2]\, y$

(iv) $D (au) = a \cdot D(u);\ D^n (au) = a \cdot D^n (u)$

(v) $D (y_1 + y_2) = D (y_1) + D(y_2);\ D^n (y_1 + y_2) = D^n (y_1) + D^n (y_2)$.

1.5 LINEAR DIFFERENTIAL EQUATION ϕ (D) y = 0

Consider $\phi(D)\, y = 0$ … (4)

where, $\phi(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + a_3 D^{n-3} + \ldots + a_{n-1} D + a_n$ is n^{th} order polynomial in D and D obeys the laws of algebra, we can in general factorise $\phi(D)$ in n linear factors as $\phi(D) = (D - m_1)(D - m_2)(D - m_3) \ldots (D - m_n)$ where $m_1, m_2, m_3, \ldots m_n$ are the roots of the algebraic equation $\phi(D) = 0$

$\therefore$ Equation (4) can be written as

$\phi(D)\, y = (D - m_1)(D - m_2)(D - m_3) \ldots (D - m_n)\, y = 0$ … (5)

Note : These factors can be taken in any sequence.

1.6 AUXILIARY EQUATION (A.E.)

The equation $\phi(D) = 0$ is called as an *auxiliary equation* (A.E.) for equations (3), (4).

e.g. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

By using operator D for $\frac{d}{dx}$, we have $(D^2 - 5D + 6)\, y = 0$

$\therefore$ $\phi(D) = D^2 - 5D + 6 = 0$ is the A.E.

$\therefore$ $(D^2 - 5D + 6)\, y = (D - 3)(D - 2)\, y = (D - 2)(D - 3)\, y$.

1.7 SOLUTION OF ϕ(D) y = 0

Being n^{th} order DE, equation (4) or (5) will have exactly n arbitrary constants in its general solution.

The equation (5) will be satisfied by the solution of the equation $(D - m_n)\, y = 0$

i.e. $\frac{dy}{dx} - m_n y = 0$

On solving this 1st order 1st degree DE by separting variables, we get $y = c_n e^{m_n x}$, where, c_n is an arbitrary constant.

Similarly, since the factors in equation (5) can be taken in any order, the equation will be satisfied by the solution of each of the equations $(D - m_1)\, y = 0$, $(D - m_2)\, y = 0$ … etc., that is by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ ………… etc.

It can, therefore, easily be proved that the sum of these individual solutions, i.e.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \qquad \ldots (6)$$

also satisfies the equation (5) and as it contains n arbitrary constants, and the equation (4) is of the n^{th} order, (6) constitutes the general solution of the equation (4).

$\therefore$ **The general solution of the equation ϕ(D) y = 0 is**

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$

where $m_1, m_2, \ldots m_n$ are the roots of the auxiliary equation $\phi(D) = 0$.

Ex. 1 : Solve $\frac{d^3 y}{dx^3} - 6\frac{d^2 y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$

Sol. : Let D stand for $\frac{d}{dx}$ and the given equation can be written as

$(D^3 - 6D^2 + 11D - 6)\, y = 0.$

Here auxiliary equation is $D^3 - 6D^2 + 11D - 6 = 0$

i.e. $(D - 1)(D - 2)(D - 3) = 0 \Rightarrow m_1 = 1, m_2 = 2, m_3 = 3$, are roots of AE.

$\therefore$ The general solution is $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$.

2. For $(4D^2 - 8D + 1)\, y = 0$, $D = 1 \pm \frac{\sqrt{3}}{2} \Rightarrow y = c_1 e^{\left(1 + \frac{\sqrt{3}}{2}\right)x} + c_2 e^{\left(1 - \frac{\sqrt{3}}{2}\right)x}$.

1.8 DIFFERENT CASES DEPENDING UPON THE NATURE OF ROOTS OF THE AUXILIARY EQUATION ϕ(D) = 0.

A. The Case of Real and Different Roots :

If roots of $\phi(D) = 0$ be $m_1, m_2, m_3 \ldots m_n$, all are real and different, then the solution of $\phi(D)\, y = 0$ will be

$$\mathbf{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}}$$

B. The Case of Real and Repeated Roots (The Case of Multiple Roots) :

Let $m_1 = m_2, m_3, m_4 \dots m_n$ be the roots of $\phi(D) = 0$, then the part of solution corresponding to m_1 and m_2 will look like

$c_1 e^{m_1x} + c_2 e^{m_1x} (m_1 = m_2) = (c_1 + c_2) e^{m_1x} = c'e^{m_1x}$

But this means that number of arbitrary constants now in the solution will be n – 1 instead of n. Hence it is no longer the general solution. The anomaly can be rectified as under.

Pertaining to $m_1 = m_2$, the part of the equation will be $(D - m_1)(D - m_1) y = 0$

Put $(D - m_1) y = z$, temporarily, then we have $(D - m_1) z = 0 \quad \therefore \; z = c_1 e^{m_1x}$

Hence putting value of z in $(D - m_1) y = z$, we have

$$(D - m_1) y = c_1 e^{m_1x} \quad \text{or} \quad \frac{dy}{dx} - m_1 y = c_1 e^{m_1x}$$

which is a linear differential equation. Its I.F. $= e^{-\int m_1 dx} = e^{-m_1x}$ and hence solution is

$$y\left(e^{-m_1x}\right) = \int c_1 e^{m_1x} \cdot e^{-m_1x} \, dx + c_2 = c_1 x + c_2$$

$$\therefore \qquad y = (c_1 x + c_2) e^{m_1x}$$

If $m_1 = m_2$ are real, and the remaining roots $m_3, m_4, m_5, \dots, m_n$ are real and different then solution of $\phi(D) y = 0$ is

$$\boxed{\mathbf{y = (c_1 x + c_2) e^{m_1x} + c_3 e^{m_3x} + c_4 e^{m_4x} + \dots + c_n e^{m_nx}}}$$

Similarly, when three roots are repeated. i.e. if $m_1 = m_2 = m_3$ are real, and the remaining roots $m_4, m_5, \dots m_n$ are real and different then solution of $\phi(D) y = 0$ is

$$\boxed{\mathbf{y = (c_1 x^2 + c_2 x + c_3) e^{m_1x} + c_4 e^{m_4x} + \dots + c_n e^{m_nx}}}$$

If $m_1 = m_2 = m_3 = \dots = m_n$ i.e. n roots are real and equal then solution of $\phi(D) y = 0$ is

$$\boxed{\mathbf{y = (c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_{n-1} x + c_n) e^{m_1x}}}$$

Ex. 1. For $(D^2 - 6D + 9) y = 0$ A.E. $= (D - 3)^2 = 0$ and solution is $y = (c_1 x + c_2) e^{3x}$

2. For $(D - 1)^3 (D + 1) y = 0$, solution is $y = (c_1 x^2 + c_2 x + c_3) e^x + c_4 e^{-x}$

3. For $(D - 1)^2 (D + 1)^2 y = 0$, solution is $y = (c_1 x + c_2) e^x + (c_3 x + c_4) e^{-x}$.

C. The Case of Imaginary (Complex) Roots

For practical problems in engineering, this case has special importance. Since the coefficients of the auxiliary equation are real, the imaginary roots (if exists) will occur in conjugate pairs. Let $\alpha \pm i\beta$ be one such pair. Therefore $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

The corresponding part of the solution of the equation $\phi(D) y = 0$, then takes the form

$$y = A e^{(\alpha + i\beta) x} + B e^{(\alpha - i\beta) x}$$

$$= e^{\alpha x} \left[A e^{i\beta x} + B e^{-i\beta x}\right]$$

$$= e^{\alpha x} [A (\cos \beta x + i \sin \beta x) + B (\cos \beta x - i \sin \beta x)]$$
$$= e^{\alpha x} [(A + B) \cos \beta x + i (A - B) \sin \beta x]$$

$$\boxed{y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]}$$

where, $c_1 = A + B$ and $c_2 = i (A - B)$ are arbitrary constants.

Using $c_1 = C \cos \theta$, $c_2 = - \sin \theta$, this can also be put sometimes into the form as given below (recall SHM).

$$\boxed{y = C e^{\alpha x} \cos (\beta x + \theta) \text{ where } C, \theta \text{ are arbitrary constants.}}$$

ILLUSTRATIONS

Ex. 1 : *Solve* $(D^2 + 2D + 5) y = 0.$

Sol. : The auxiliary equation is $D^2 + 2D + 5 = 0$ whose roots are $D = - 1 \pm 2i$ which are both imaginary. Here $\alpha = - 1$, $\beta = 2$. Hence the solution is

$$y = e^{-x} [A \cos 2x + B \sin 2x]$$

Ex. 2 : *Solve* $\dfrac{d^4y}{dx^4} - 5 \dfrac{d^2y}{dx^2} + 12 \dfrac{dy}{dx} + 28y = 0.$

Sol. : The auxiliary equation is $D^4 - 5D^2 + 12D + 28 = 0$ having roots $D = - 2, -2, 2 \pm \sqrt{3}\, i$.

(Here $\alpha = 2$, $\beta = \sqrt{3}$). Hence the solution is

$$y = (c_1 x + c_2) e^{-2x} + e^{2x} \left[A \cos \sqrt{3}\, x + B \sin \sqrt{3}\, x\right]$$

Ex. 3 : For $(D^2 + 4)y = 0$, $D = 0 \pm 2i$ (Here $\alpha = 0$, $\beta = 2$) $\Rightarrow$ $y = A \cos 2x + B \sin 2x$.

D. The Case of Repeated Imaginary Roots :

If the imaginary roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ occur twice, then the part of solution of $\phi (D) y = 0$ will be

$$y = (A x + B) e^{m_1 x} + (C x + D) e^{m_2 x} \qquad \text{… (by using case B)}$$
$$= (A x + B) e^{(\alpha + i\beta) x} + (C x + D) e^{(\alpha - i\beta) x}$$
$$= e^{\alpha x} \left[(A x + B) e^{i\beta x} + (C x + D) e^{-i\beta x}\right]$$
$$= e^{\alpha x} [(A x + B) \{\cos \beta x + i \sin \beta x\} + (C x + D) \{\cos \beta x - i \sin \beta x\}]$$
$$= e^{\alpha x} [(A x + B + C x + D) \cos \beta x + i (A x + B - C x - D) \sin \beta x]$$

$$\boxed{y = e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]}$$

with proper changes in the constants c_1, c_2, c_3 and c_4.

ILLUSTRATIONS

Ex. 1 : *Solve* $\frac{d^6y}{dx^6} + 6\frac{d^4y}{dx^4} + 9\frac{d^2y}{dx^2} = 0.$

Sol. : The auxiliary equation $D^6 + 6D^4 + 9D^2 = 0$ has roots $D = 0, 0, \pm i\sqrt{3}, \pm i\sqrt{3}$ where the imaginary roots $\pm i\sqrt{3}$ are repeated. Hence the solution is

$$y = c_1 x + c_2 + (c_3 x + c_4) \cos \sqrt{3}\, x + (c_5 x + c_6) \sin \sqrt{3}\, x$$

Ex. 2 : $(D^4 + 2D^2 + 1)\, y = 0.$

Sol. : The auxiliary equation $D^4 + 2D^2 + 1 = 0$ has roots $D = \pm i, \pm i$, repeated imaginary roots. Hence the solution is

$$y = (c_1x + c_2) \cos x + (c_3x + c_4) \sin x$$

Now we will summarise the four cases for ready reference.

Case 1 : Real & Distinct Roots : A.E. $\Rightarrow (D - m_1)(D - m_2)(D - m_3) \ldots (D - m_n) = 0$ $\therefore$ **Solution is** $y = c_1 e^{m_1x} + c_2 e^{m_2x} + c_3 e^{m_3x} + \ldots + c_n e^{m_nx}$
Case 2 : Repeated Real Roots :   For $m_1 = m_2 \Rightarrow$ A.E. $\Rightarrow (D - m_1)(D - m_1)(D - m_3) \ldots\ldots (D - m_n) = 0$ **Solution is** $y = (c_1 x + c_2) e^{m_1x} + c_3 e^{m_3x} + \ldots + c_n e^{m_nx}$ For $m_1 = m_2 = m_3 \Rightarrow$ A.E. $\Rightarrow (D - m_1)(D - m_1)(D - m_1)(D - m_4) \ldots (D - m_n) = 0$ **Solution is** $y = (c_1 x^2 + c_2 x + c_3) e^{m_1x} + c_4 e^{m_4x} + \ldots + c_n e^{m_nx}$
Case 3 : Imaginary Roots : For $D = \alpha \pm i\beta$ **Solution is** $y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$
Case 4 : Repeated Imaginary Roots : For $D = \alpha \pm i\beta$ be repeated twice   **Solution is** $y = e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]$

ILLUSTRATIONS

1. Solve $\frac{d^2 x}{dt^2} + 4x = 0$. Let D stand for $\frac{d}{dt}$.

$\therefore$ A.E. : $D^2 + 4 = 0 \Rightarrow D = 0 \pm 2i$

$\therefore$ The solution is $x = c_1 \cos 2t + c_2 \sin 2t$.

2. Solve $\frac{d^4 y}{dz^2} - 16y = 0$. Let D stand for $\frac{d}{dz}$.

$\therefore$ A.E. : $D^4 - 16 = 0$, $(D - 2)(D + 2)(D^2 + 4) = 0$.

$\therefore$ The solution is $y = c_1 e^{2z} + c_2 e^{-2z} + c_3 \cos 2z + c_4 \sin 2z$.

Special Case : If the two real roots of $\phi(D)\, y = 0$ be m and – m [e.g. $D^2 - m^2 = 0$], then the corresponding part of the solution is