As per new JNTU, Hyderabad syllabus

ENGINEERING MATHEMATICS-III

(Recommended for B.Tech. Second Year Students)

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ENGINEERING MATHEMATICS

VOLUME III

(For JNTU B.Tech. Second Year, First Semester Students of ECE, EEE and Allied Branches)

(Second Year, Second Semester Students of Metallurgy & Material Engg. and Petroleum Engg.)

RECOMMENDED FOR
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JNTU, HYDERABAD

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PREFACE TO THE ELEVENTH REVISED EDITION

We are happy to bring out the Eleventh revised edition of the book "*Engineering Mathematics*, *Vol. III*". The earlier editions have been received well by the student and teacher community. This textbook has been written strictly according to the revised syllabus 2013-14 (R13) effective from 2014 - 15 of **B. Tech. II Year, First Semester** students of Jawaharlal Nehru Technological University, **Hyderabad**. This edition has been thoroughly reviewed in light of the latest syllabus.

A good number of worked examples have been added and questions from latest university question papers have been included at appropriate places.

The authors are grateful to Sri B.V.S.S. Sarma, author of Intermediate and Degree Textbooks on Mathematics for the help rendered by him at various stages of planning of this book.

All suggestions for further improvement of the book will be received thankfully by us, to serve the cause of imparting good, correct and latest information to the students.

We are thankful to the Management Team and the Editorial Department of S. Chand & Company Pvt. Ltd., New Delhi for all help and support in the publication of this book.

— AUTHORS

SYLLABUS 2014 - 2015 (R13)

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY

HYDERABAD

II Year B.Tech. I Semester

MATHEMATICS–III

UNIT - I : Linear ODE with variable coefficients and series solutions (second order only) :

Equations reducible to constant coefficients - Cauchy's and Lagrange's differential equations. Motivation for series solutions, Ordinary point and Regular singular point of a differential equation, Transformation of non-zero singular point to zero singular point. Series solutions to differential equations around zero, Frobenius Method about zero.

UNIT - II : Special Functions :

Legendre's Differential equation, General solution of Legendre's equation, Legendre polynomials properties : Rodrigue's formula - Recurrence relations, Generating function of Legendre's polynomials - Orthogonality. Bessel's Differential equation, Bessel functions properties : Recurrence relations, Orthogonality, Generating function, Trigonometric expansions involving Bessel functions.

UNIT - III : Complex Functions - Differentiation and Integration :

Complex functions and its representation on Argand plane, Concepts of limit, continuity, Differentiability, Analyticity, Cauchy-Riemann conditions, Harmonic functions - Milne - Thompson method. Line integral - Evaluation along a path and by indefinite integration - Cauchy's integral theorem - Cauchy's integral formula - Generalized formula.

UNIT - IV : Power series expansions of complex functions and contour Integration :

Radius of convergence- Expansion in Taylor's series, Maclaurin's series and Laurent series. Singular point- Isolated singular point-pole of order m-essential singularity. Residue - Evaluation of residue by formula and by Laurent series - Residue theorem. Evaluation of integrals of the type:

(a) Improper real integrals $\int_{-\infty}^{\infty} f(x)\,dx$ (b) $\int_{c}^{c+2\pi} f(\cos\theta, \sin\theta)\,d\theta$

UNIT - V : Conformal mapping :

Transformation of z - plane to w-plane by a function, Conformal transformation. Standard transformations - Translation; Magnification and rotation; Inversion and reflection, Transformations like e^z, $\log z$, z^2 and Bilinear transformation. Properties of Bilinear transformation, determination of bilinear transformation when mappings of 3 points are given.

REFERENCES :

Engineering Mathematics, Volume - III - T.K.V. Iyengar, B. Krishna Gandhi, S. Ranganatham and M.V.S.S.N. Prasad, S. Chand and Company.

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UNIT I

Chapter - 1

Linear Ordinary Differential Equations with Variable Coefficients

Chapter - 2

Series Solutions of Second order Differential Equations

CHAPTER 1

Linear Ordinary Differential Equations with Variable Coefficients

1.1 INTRODUCTION

We have already studied how to find the solution of a differential equation with constant coefficients.

Now we shall study linear differential equations with variable coefficients.

In this chapter we will discuss two forms of linear differential equations with variable coefficients namely Cauchy's and Lagrange's differential equations which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

1.2 CAUCHY'S HOMOGENEOUS LINEAR EQUATION

An equation of the form $x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots.. + P_{n-1} x \frac{dy}{dx} + P_n y = \phi(x)$ ... (1)

where $P_1, P_2, \ldots.., P_n$ are real constants and $\phi(x)$ is a function of x is called Cauchy's Homogeneous linear equation or Euler–Cauchy's linear equation of order n.

The equation in the operator form is $(x^n D^n + P_1 x^{n-1} D^{n-1} + \ldots. + P_{n-1} xD + P_n) y = \phi(x)$

where $\frac{d}{dx} = D$. Cauchy's linear differential equation can be transformed into a linear equation with constant coefficients by the change of independent variable with the substitution

$$x = e^z \text{ (or) } z = \log x, \; x > 0$$

$$\therefore \frac{dz}{dx} = \frac{1}{x}. \quad \text{Now} \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

or $$x \cdot \frac{dy}{dx} = \frac{dy}{dz} \quad \ldots(2)$$

Again $$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$ [using Product Rule]

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)$$ [Multiply and divide by dz]

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \quad \left[\because \frac{dz}{dx} = \frac{1}{x}\right]$$

i.e., $x^2 \dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}$ ...(3)

Similarly, we can prove that

$x^3 \dfrac{d^3y}{dx^3} = \dfrac{d^3y}{dz^3} - 3\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz}$ ...(4)

Let us denote $\dfrac{d}{dx} \equiv D$ and $\dfrac{d}{dz} \equiv \theta$. Then (2), (3), (4) can be written as

$xD = \theta, \quad x^2D^2 = \theta(\theta - 1), x^3D^3 = \theta(\theta - 1)(\theta - 2)$ etc.

Substituting (2), (3), (4) and so on in (1), the Euler–Cauchy equation reduces to a differential equation with constant coefficients where y is dependent variable and z is independent variable. By methods discussed earlier, the equation can be solved and we get the required solution by putting $z = \log x$ in the obtained solution.

This will be illustrated through the following examples.

SOLVED EXAMPLES

Example 1 : Solve $(x^2D^2 - 4xD + 6)y = x^2$

Solution : Given equation is $(x^2D^2 - 4xD + 6)y = x^2$ ...(1)

This is a homogeneous differential equation.

Let $x = e^z$. Then $\log x = z$ ...(2)

Let $\dfrac{d}{dx} = D$ and $\dfrac{d}{dz} = \theta$. Then we have $xD = \theta$ and $x^2D^2 = \theta(\theta - 1)$

Substituting in (1), we get

$$[\theta(\theta - 1) - 4\theta + 6]y = e^{2z} \text{ or } (\theta^2 - 5\theta + 6)y = e^{2z} \quad ...(3)$$

This is a differential equation with constant coefficients.

A.E. is $m^2 - 5m + 6 = 0$. The roots are $m = 3$ and $m = 2$ which are real and different.

Hence C.F. is $y_c = c_1e^{3z} + c_2e^{2z}$

Now P.I. $= y_p = \dfrac{e^{2z}}{(\theta - 3)(\theta - 2)} = \dfrac{e^{2z}}{(2-3)}\dfrac{z}{1!} = -ze^{2z}$

$\therefore$ General solution is $y = y_c + y_p \Rightarrow y = c_1e^{3z} + c_2e^{2z} - ze^{2z}$

$\Rightarrow y = c_1x^3 + c_2x^2 - (\log x)x^2$

Example 2 : Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x$ **[JNTU 1995, 2006 (Set No.4)]**

Solution : Given equation in the operator form is

$$(x^2D^2 - xD + 1)y = \log x, \text{ where } D \equiv \frac{d}{dx} \quad ...(1)$$

Let $x = e^z$ so that $z = \log x$ ...(2)

Denoting $\theta = d/dz$, $xD = \theta, x^2D^2 = \theta(\theta-1)$, so that (1) becomes

$$\left[\theta(\theta-1)-\theta+1\right]y = z$$

i.e. $(\theta^2-2\theta+1)y = z$ ...(3)

This is a linear differential equation with constant coefficients.

A.E. is $m^2-2m+1=0 \Rightarrow (m-1)^2=0.$

$\therefore m = 1, 1$

$\therefore$ The roots are real and equal.

$\therefore$ C.F. $= y_c = (c_1+c_2z)e^z$

$$\text{P.I.} = y_p = \frac{z}{(\theta-1)^2} = (1-\theta)^{-2}(z) = (1+2\theta+....)z = z+2$$

$\therefore$ General solution of (1) is $y = y_c + y_p$

i.e. $y = (c_1+c_2z)e^z+(z+2) = (c_1+c_2\log x)x+\log x+2$

Example 3 : Solve $x^3\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+2y = 10\left(x+\dfrac{1}{x}\right)$. **[JNTU 2007S (Set No. 2)]**

Solution : Given equation in the operator form is

$(x^3D^3+2x^2D^2+2)y = 10\left(x+\dfrac{1}{x}\right)$ where $\dfrac{d}{dx} \equiv D.$ ...(1)

Let $x = e^z$ or $\log x = z$ ...(2)

Denote $\dfrac{d}{dz} \equiv \theta$. Then we have

$x^3D^3 = \theta(\theta-1)(\theta-2)$ ...(3)

and $x^2D^2 = \theta(\theta-1)$ ...(4)

Substituting in (1), we get

$$\left[\theta(\theta-1)(\theta-2)+2\theta(\theta-1)+2\right]y = 10(e^z+e^{-z})$$

i.e. $(\theta^3-\theta^2+2)y = 10(e^z+e^{-z})$ ...(5)

This is a linear differential equation with constant coefficients.

Let $f(\theta) = \theta^3-\theta^2+2$. Then A.E. is $f(m) = 0$ *i.e.*, $m^3-m^2+2 = 0$

$(m+1)\ (m^2-2m+2) = 0$. The roots are $m = -1,\ 1+i,\ 1-i.$

One root is real and two roots are complex conjugate numbers.

$\therefore$ C.F. $= y_c = c_1e^{-z}+e^z(c_2\cos z+c_3\sin z)$ and P.I. $= \dfrac{10(e^z+e^{-z})}{\theta^3-\theta^2+2} = y_{p_1}+y_{p_2}$

Now $y_{p_1} = \text{PI}_1 = 10\left(\dfrac{e^z}{\theta^3-\theta^2+2}\right)$ [Put $\theta = 1$]

$$= 10\left(\frac{e^z}{1-1+2}\right) = 5e^z$$

and $y_{p_2} = \text{PI}_2 = 10\left(\frac{e^{-z}}{\theta^3 - \theta^2 + 2}\right) = 10\frac{(e^{-z})}{(\theta+1)(\theta^2 - 2\theta + 2)}$ (case of failure)

$$= \frac{10(e^{-z})}{5} z = 2ze^{-z}$$

The general solution is $y = y_c + y_{p1} + y_{p2}$

i.e. $y = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z) + 5e^z + 2ze^{-z}$

or $y = c_1 x^{-1} + x(c_1 \cos \log x + c_2 \sin \log x) + 5x + 2\log x\left(\frac{1}{x}\right)$

Example 4 : Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$ **[JNTU 2003S (Set No. 3)]**

Solution : Given equation can be written as $(x^2D^2 - 3xD + 4)y = (1+x)^2$ ...(1)

where $\frac{d}{dx} = D.$ This is a homogeneous linear differential equation.

Let $x = e^z$ or $\log x = z$

If $\frac{d}{dz} = \theta$ then we have $xD = \theta$; $x^2D^2 = \theta(\theta - 1)$

Substituting in (1), we get $[\theta(\theta-1) - 3\theta + 4]y = (1+e^z)^2$.

i.e., $(\theta^2 - 4\theta + 4)y = 1 + e^{2z} + 2e^z$ ...(2)

This is a differential equation with constant coefficients.

AE is $m^2 - 4m + 4 = 0$

i.e., $(m-2)^2 = 0 \Rightarrow m = 2, 2$

Roots are real and equal.

$\therefore$ C.F. is $y_c = (c_1 + c_2 z)e^{2z}$

$$\text{P.I.} = \frac{1 + e^{2z} + 2e^z}{\theta^2 - 4\theta + 4} = \frac{1}{\theta^2 - 4\theta + 4} + \frac{e^{2z}}{\theta^2 - 4\theta + 4} + \frac{2e^z}{\theta^2 - 4\theta + 4}$$

$$y_{p_1} = \text{PI}_1 = \frac{1}{\theta^2 - 4\theta + 4} = \frac{e^{0.z}}{\theta^2 - 4\theta + 4} \quad [\text{Put } \theta = 0]$$

$$\therefore \quad \text{PI}_1 = \frac{1}{4}$$

$$y_{p_2} = \text{PI}_2 = \frac{1}{\theta^2 - 4\theta + 4}(e^{2z}) = e^{2z}\left(\frac{z^2}{2}\right) \quad [\because f(a) = 0]$$

$$y_{p_3} = \text{PI}_3 = \frac{2}{\theta^2 - 4\theta + 4}(e^z) = \frac{2e^z}{(1)^2 - 4(1) + 4} = 2e^z$$

$\therefore$ General solution is $y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

i.e., $y = (c_1 + c_2 z)e^{2z} + \frac{1}{4} + e^{2z}\frac{z^2}{2} + 2e^z$

$$= (c_1 + c_2 \log x)x^2 + \frac{1}{4} + x^2 \frac{(\log x)^2}{2} + 2x$$

Example 5 : Solve $\left(x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8\right)y=65\cos(\log x)$

Solution : Given equation in the operator form is

$$(x^3D^3+3x^2D^2+xD+8)y=65\cos(\log x) \quad ...(1)$$

where $\frac{d}{dx}\equiv D.$ Put $x=e^z$ or $\log x=z$ ...(2)

Denote $\frac{d}{dz}\equiv\theta$, then we have

$$x^3D^3=\theta(\theta-1)(\theta-2) \quad ...(3)$$

$$x^2D^2=\theta(\theta-1) \quad ...(4)$$

$$xD=\theta \quad ...(5)$$

Substituting in (1), we get $[\theta(\theta-1)(\theta-2)+3\theta(\theta-1)+\theta+8]y=65\cos z$

i.e. $(\theta^3+8)y=65\cos z$ ...(6)

This is a linear differential equation with constant coeffcients.

A.E. is $m^3+8=0 \Rightarrow (m+2)(m^2-2m+4)=0$ $\left[\because \boldsymbol{a^3+b^3=(a+b)(a^2-ab+b^2)}\right]$

$\therefore\ m=-2, m=1+i\sqrt{3}$ and $m=1-i\sqrt{3}$

One root is real and other two roots are complex conjugate numbers.

C.F. $=y_c=c_1e^{-2z}+e^z(c_2\cos\sqrt{3}z+c_3\sin\sqrt{3}z)$

P.I. $=y_p=\frac{65\cos z}{\theta^3+8}=\frac{65\cos z}{-\theta+8}=\frac{65(8+\theta)\cos z}{64-\theta^2}$ [Put $\theta^2=-1$]

$=\frac{65(8+\theta)\cos z}{64+1}=8\cos z-\sin z$

General solution is $y=y_c+y_p$

i.e. $y=c_1e^{-2z}+e^z(c_2\cos z\sqrt{3}+c_3\sin z\sqrt{3})+8\cos z-\sin z$

or $y=c_1x^{-2}+x\left[c_2\cos(\sqrt{3}\log x)+c_3\sin(\sqrt{3}\log x)\right]+8\cos(\log x)-\sin(\log x).$

Example 6 : Solve $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$ **[JNTU 1993]**

Solution : Given equation in the operator form is

$$(x^2D^2-xD+2)y=x\log x \quad ...(1)$$

where $\frac{d}{dx}\equiv D$. Put $x=e^z$ so that $z=\log x$ ...(2)

Denote $\frac{d}{dz}\equiv\theta$. Then we have

$$x^2D^2=\theta(\theta-1) \quad ...(3)$$

$$xD=\theta \quad ...(4)$$

Substituting in (1), we get $[\theta(\theta-1)-\theta+2]y=z\,e^z$

i.e. $(\theta^2-2\theta+2)y=ze^z$ ...(5)

This is a linear differential equation with constant coeffcients.

AE is $m^2 - 2m + 2 = 0$. The roots are $1 \pm i$ which are complex conjugate numbers.

$\therefore$ C.F $= y_c = e^z(c_1 \cos z + c_2 \sin z)$

$$\text{P.I.} = y_p = \frac{ze^z}{\theta^2 - 2\theta + 2} = e^z \frac{z}{(\theta+1)^2 - 2(\theta+1) + 2}$$

$$= e^z \frac{z}{(\theta^2+1)} = e^z(1+\theta^2)^{-1}(z) = e^z(1 - \theta^2 + \ldots)z = ze^z$$

The general solution is $y = y_c + y_p$

i.e. $y = e^z(c_1 \cos z + c_2 \sin z) + ze^z$

or $y = x[c_1 \cos(\log x) + c_2 \sin(\log x)] + x \log x.$

Example 7 : Solve $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$

Solution : Given equation is $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$ ...(1)

Let $x = e^z$ so that $z = \log x$ ...(2)

Denote $\dfrac{d}{dz} = \theta$, then we have $x^2D^2 = \theta(\theta - 1), xD = \theta$ ...(3)

Substituting in (1), we get ...(4)

$$[\theta(\theta-1) + 3\theta + 1]y = \frac{1}{(1-e^z)^2}$$

i.e. $(\theta^2 + 2\theta + 1)y = \dfrac{1}{(1-e^z)^2}$ ...(5)

This is a differential equation with constant coefficients.

AE. is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$ $\therefore$ The roots are $m = -1, -1$

The roots are real and equal .

C.F. $= y_c = (c_1 + c_2 z)e^{-z}$

$$\text{P.I.} = \frac{1}{(\theta+1)^2(1-e^z)^2} = \frac{1}{\theta+1}\left[\frac{1}{\theta+1} \cdot \frac{1}{(1-e^z)^2}\right]$$

$$= \frac{1}{\theta+1}\left[e^{-z}\int \frac{1}{(1-e^z)^2} e^z dz\right] = \frac{1}{\theta+1}\left[e^{-z}\frac{1}{1-e^z}\right]$$

$$= e^{-z}\int \frac{e^{-z}}{1-e^z} e^z dz = e^{-z}\int \frac{dz}{1-e^z} = e^{-z}\int \frac{e^{-z}}{e^{-z}-1} dz = -e^{-z}\log(e^{-z}-1).$$

The general solution is given by $y = y_c + y_p$

i.e. $y = (c_1 + c_2 z)e^{-z} - e^{-z}\log(e^{-z} - 1)$

$$= (c_1 + c_2 \log x)\frac{1}{x} - \frac{1}{x}\log\left(\frac{1}{x} - 1\right)$$

Engineering Mathematics III