Eastern Economy Edition

Abstract Algebra

Second Edition

I.H. Sheth

PHI

Abstract Algebra

Second Edition

I.H. SHETH
(Retired)
Department of Mathematics
University School of Sciences
Gujarat University
Ahmedabad

PHI Learning Private Limited
New Delhi-110001
2009

ABSTRACT ALGEBRA, 2nd ed.
I.H. Sheth

ISBN-978-81-203-3932-3

Fourth Printing (Second Edition) ... ... October, 2009

Published by Asoke K. Ghosh, PHI Learning Private Limited, M-97, Connaught Circus, New Delhi-110001 and Printed by Meenakshi Printers, Delhi-110006.

CONTENTS

Preface *ix*
Preface to the First Edition *xi*
Note to the Student *xiii*
Notations *xv*

Part I: PRELIMINARIES

1. SET THEORY **3–9**

1.1 Introduction 3
1.2 Set Theoretic Ideas 3
1.3 Set Operations—Union, Intersection, and Symmetric Difference 6
1.4 Partition of a Set 8
1.5 Product Set and Power Set 9
Exercises 9

2. RELATIONS **10–18**

2.1 Definitions and Examples 10
2.2 Equivalence Relation 12
2.3 Importance of the Equivalence Relation 16
2.4 Solved Examples 16
Exercises 17

3. MAPPINGS **19–31**

3.1 Definitions and Examples 19
3.2 Properties: One-One and Onto of a Mapping 22
3.3 Composition of Mappings 23
3.4 Inverse Mapping and One-One Correspondence 26
3.5 Solved Examples 29
Exercises 30

4. MATRICES **32–56**

4.1 Definitions and Examples 32
4.2 Algebraic Operations for Matrices 34
4.3 Properties of Operations—Addition and Multiplication 38
4.4 Special Types of Matrices 45
4.5 Solved Examples 50
Exercises 54

5. BINARY OPERATIONS **57–74**

5.1 Definitions and Examples 57
5.2 Properties of Binary Operations 62
5.3 Identity Element and Inverse Element 65
5.4 Cancellation Law 68
5.5 Solved Examples 69
Exercises 71

6. INTEGERS **75–91**

6.1 Introduction 75
6.2 Division Algorithm 76
6.3 Greatest Common Divisor and Least Common Multiple 78
6.4 Prime Numbers 81
6.5 Congruence Relation 83
6.6 Solved Examples 89
Exercises 90

Part II: GROUP THEORY

7. GROUPS **95–124**

7.1 Introduction 95
7.2 Definitions and Examples 97
7.3 Elementary Properties of a Group 102
7.4 Equivalent Definitions of a Group 103
7.5 Generalized Form of Associative Law 105
7.6 Finite Groups and Their Tables 112
7.7 Solved Examples 117
Exercises 120

8. SUBGROUPS **125–142**

8.1 Definitions and Examples 125
8.2 Lagrange's Theorem 129
8.3 Solved Examples 134
Exercises 137

9. PERMUTATIONS **143–160**

9.1 Definitions and Examples 143
9.2 Transposition and Cycle 146
9.3 Solved Examples 157
Exercises 159

10. NORMAL SUBGROUPS **161–171**

10.1 Definitions and Examples 161
10.2 Quotient Group 164
10.3 Solved Examples 167
Exercises 170

11. ISOMORPHISM OF GROUPS **172–181**

11.1 Definitions and Examples 172
11.2 Solved Examples 176
Exercises 179

12. CYCLIC GROUPS **182–189**

12.1 Properties of Cyclic Groups 182
12.2 Isomorphisms of Cyclic Groups 184
12.3 Subgroups of a Cyclic Group 184
12.4 Generators of a Cyclic Group 186
12.5 Solved Examples 187
Exercises 189

13. HOMOMORPHISM: 1 **190–205**

13.1 Definitions and Examples 190
13.2 Kernel of Homomorphism 193
13.3 Cayley's Theorem 196
13.4 Isomorphism of Groups 198
13.5 Groups of Order Four and Six 199
13.6 Solved Examples 202
Exercises 203

Part III: RING THEORY

14. RINGS **209–227**

14.1 Definitions and Examples 209
14.2 Properties of a Ring 215
14.3 Further Examples of Rings 217
14.4 Solved Examples 220
Exercises 224

15. INTEGRAL DOMAINS **228–236**

15.1 Zero-divisor and Definitions 228
15.2 Characteristic of a Ring 232
15.3 Solution of the Equation $ax = b$ in a ring R 233
15.4 Solved Examples 234
Exercises 235

16. IDEALS: 1 **237–250**

16.1 Subrings 237
16.2 Ideals 239
16.3 Quotient Ring 243
16.4 Solved Examples 245
Exercises 248

17. HOMOMORPHISM: 2 **251–258**

17.1 Definitions and Examples 251
17.2 Homomorphism and Characteristic 255
17.3 Solved Examples 256
Exercises 257

18. FIELDS **259–267**

18.1 Extension Field 259
18.2 The Field of Quotients of an Integral Domain 260
18.3 Prime Fields 265
18.4 Solved Examples 266
Exercises 267

19. POLYNOMIALS **268–311**

19.1 Introduction 268
19.2 Integral Domain $D[x]$ 270
19.3 Familiar Form of Integral Domain $D[x]$ 277
19.4 Unique Factorization of Polynomials 281
19.5 Solutions of a Polynomial Equation 293
19.6 Eisenstein Criterion for Irreducibility 299
19.7 Solved Examples 303
Exercises 306

20. IDEALS: 2 **312–321**

20.1 Introduction 312
20.2 Maximal Ideal 312
20.3 Prime Ideal 314
20.4 Solved Examples 316
Exercises 320

21. FINITE FIELDS **322–329**

21.1 Introduction 322
21.2 Finite Fields 326
21.3 Construction of a Field with Four Elements 327
21.4 Solved Examples 328
Exercises 329

22. UNIQUE FACTORIZATION DOMAIN AND EUCLIDEAN RING **330–350**

22.1 Introduction 330
22.2 Divisibility 330
22.3 Unit Elements and Associate Elements 331
22.4 Greatest Common Divisor 333
22.5 Prime and Composite Elements 334
22.6 Unique Factorization Domain 334
22.7 Principal Ideal Domain and Unique Factorization Domain 337
22.8 Euclidean Ring 341
22.9 Solved Examples 344
Exercises 347

Appendices ***351–355***

A: De Moivre's Theorem 351
B: Three Famous Problems 354

Supplementary Problems ***357–364***

References ***365***

PREFACE

Since the publication of the first edition of this book, several professors and students of Abstract Algebra across India have proactively reached out to me to let me know how they have benefited from this book. As an author, I am really glad to know that the book has been so well received by the teachers and undergraduate/postgraduate students of Abstract Algebra in particular, and the broader math community in general. This gives an immense sense of satisfaction to me, and naturally, I have incorporated in this second edition many suggestions and comments that I received from teachers and students.

In order to make the second edition more beneficial to the readers, I also discussed with several users of the book some of the areas where there was an opportunity to revise the first edition. Based on these discussions as well, I have made some important changes in the second edition:

1. Added a chapter that primarily focuses on matrices to enable the reader to use the knowledge acquired from it to appreciate several concepts of Abstract Algebra that are discussed throughout the book.

2. Provided proofs for a couple of theorems where such details were not available earlier; provided more detailed explanations of some the solved problems, and also moved some of the exercise problems into the category of solved examples to make the book more comprehensive.

3. As an elementary application of the subject and in particular of the permutation groups, relevant explanation of the determinant of a real matrix has been added to Chapter 9.

I hope that the changes introduced in the second edition will make the book more comprehensive and balanced for the teachers and students of Abstract Algebra at the undergraduate/postgraduate level.

I would like to express my sincere thanks to Prof. N.N. Roghelia of M.G. Science Institute, Ahmedabad for his valuable suggestions and in particular for going through the new chapter on matrices in detail. I would also like to thank and compliment PHI Learning for bringing out this second edition in such a presentable form. People I worked with at PHI were very well organized and highly skilled professionals, and it was a joy working with them, so my special thanks to all of them.

Last but not the least, as I mentioned earlier, as an author, I felt immensely pleased to interact with the users of this book. These interactions over time have influenced the revision of the book in a positive way. I sincerely thank all the teachers and students for this wonderful cooperation, and look forward to hearing more from them again.

I.H. SHETH

PREFACE TO THE FIRST EDITION

My interest in abstract algebra began when I was assigned sessions of problem solving in a refresher course for teachers organized by M.S. University of Baroda in 1967. The refresher course was part of the programme to familiarize teachers with modern mathematics. This opportunity gave me immense confidence in the subject matter. I read a number of excellent books and found that most of them pertained to the postgraduate level. There was hardly any book suitable solely for the undergraduate students. This inspired me to write a book on abstract algebra solely for the undergraduate students of Indian universities. This book is thus an outcome of my notes prepared for orientation courses for teachers as well as regular courses for undergraduate students.

The book is divided into three parts. In Part I (Chapters 1–5), the concepts necessary for the subject are explained. In Chapter 1, union and intersection for a family of sets are defined. Equivalence relation plays an important role in modern mathematics. Chapter 2 is devoted to the study of relation, in general. The motivational factors behind the algebraic structure—group—are the properties of the set $A(S)$ of one-one correspondences defined on a nonempty set S and these ideas are studied in Chapter 3. Here, I took an opportunity to define a mapping in terms of a relation satisfying additional properties. This approach helps students understand the notion of inverse mapping of a given mapping in a natural way. Binary operations and their properties are studied in Chapter 4. Students also study here special elements with respect to a given binary operation. This general approach helps students in understanding algebraic structures, and other similar structures in linear algebra, in a better way. Chapter 5 introduces some basic elementary concepts of the number theory which are important in our study. The properties of set $\mathbb{Z}$ of integers under operations of addition and multiplication lead to the notion of another algebraic structure—ring—while the notion of prime integers leads to the parallel notions of integral domain and unique factorization domain.

In Part II (Chapters 6–12), we study our first algebraic structure—group—having only one binary operation. General properties of a group are given in Chapter 6 while the famous Lagrange's theorem is proved in Chapter 7. The permutation group S_n, the concepts of transposition and cycle and their

properties are studied in detail in Chapter 8. It becomes essential to recognize when two algebraic structures are identical (isomorphic in mathematical language). Because of this, the notion of isomorphism between two algebraic structures becomes fundamental. Chapters 10 and 12 are devoted to the study of isomorphic groups. The study of cyclic groups is very interesting. They are considered the building blocks for finite abelian groups. Chapter 11 is devoted to the study of cyclic groups.

Part III (Chapters 13–21) deals with the study of algebraic structure—ring—having two binary operations. Here we study integral domains, fields, and polynomial rings in detail. Also, the notions of an ideal, homomorphism, parallel to similar concepts in group theory, are studied. These similarities between parallel concepts will help students in the comparative study of these two algebraic structures. Some elementary properties of a Euclidean ring as well as those of a unique factorization domain are also given.

The hallmark of the book is that each important topic is discussed in a separate chapter according to its requirement. Also an important definition or notion is preceded as well as followed by a number of examples. This approach helps students to grasp abstract notions in familiar concrete forms. In addition, each chapter contains a special section—solved examples. In this section, some important properties as well as notions related with the chapter matter are also considered. The drill of solving problems reinforces the student's understanding of a concept. End-of-chapter exercises and supplementary problems are designed to achieve this objective. At the end of the book, a small list of references is provided. As mentioned in the beginning, this book is written only for undergraduate students and hence some interesting results which are beyond the scope of this book are just mentioned with proper references. I hope an enthusiastic reader will definitely read beyond the subject matter given in this book using this list.

I am extremely thankful to my colleague Dr. (Miss) Nita Shah who helped me in preparing the soft copy of the manuscript. It is a matter of pleasure to thank my wife, Yashodhara, for encouraging me throughout the preparation of the manuscript. I also thank Prentice-Hall of India Private Limited for their enthusiastic efforts in bringing out this book in an extremely presentable form.

I would welcome critical comments and suggestions from teachers and students alike for improvement of this book.

Though I started teaching abstract algebra in the very first year of my academic life, I still very much enjoy teaching this subject. I enjoyed writing this book as well. I will consider my efforts amply rewarded if my enthusiasm for abstract algebra is also transmitted to readers through this work.

I.H. SHETH

NOTE TO THE STUDENT

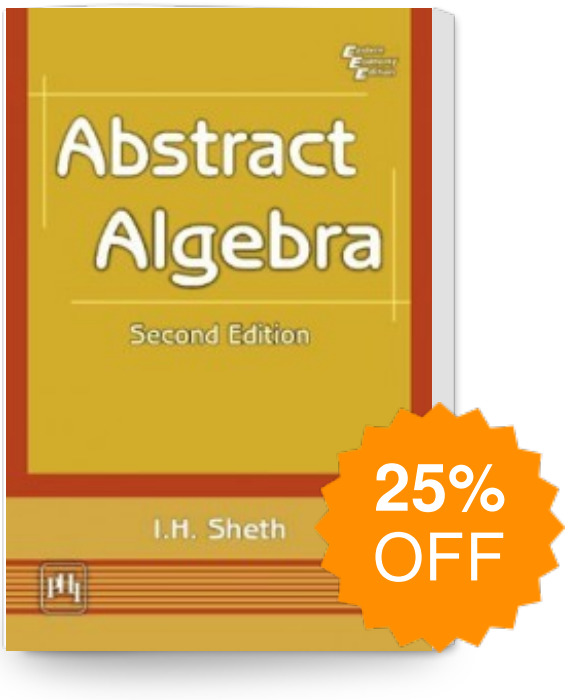
The last two centuries have seen many new branches of mathematics. Abstract algebra is one such branch. Because of its vast applications in various disciplines of mathematics and also in different subjects like physics, chemistry, statistics, and computer science, it becomes necessary to introduce this topic at the undergraduate-level curriculum.

The aim of this book is to familiarize students with the basic concepts of abstract algebra and I have taken complete care to introduce each notion with sufficient number of examples. Students are advised to go through these examples in detail and to construct new examples. Each important topic is treated in a separate chapter, taking care of the continuity of the subject matter which is very essential in a basic course.

According to me, the analysis of a proof of a theorem is more important rather than going through the entire proof in a routine way. What are the assumptions in the statement of a theorem? Where are these assumptions used in the course of a proof? In short, a proper justification for each subsequent step in a proof should be given. This helps students to learn mathematics in a true sense. I have followed this approach throughout the text. My long teaching experience has shown that students tend to prove results mechanically without going into the validity of steps. This is not the correct way to learn mathematics. I have therefore tried to reason out each step in each proof and asked the students to give their reasoning whenever required. I always encourage students to form this habit of reasoning. An enthusiastic student can go one step further and think about—what will be the conclusion under change of assumption(s) in the statement of a theorem? Will the theorem remain true under a weaker condition? Or can some more additional properties be achieved under a stronger condition? For example, if D is an integral domain then the polynomial ring $D[x]$ is also an integral domain as proved in Theorem 19.2.2. Will it remain an integral domain under the weak condition, namely if D were simply a ring? Similarly, the integral domain $D[x]$ is not a field. Will it become a field under the stronger condition, namely if D were a field? Ultimately, this habit of thinking will lead students either to search for a proof or to construct a counter example for the expected result.

The author always enjoys receiving comments from the reader. Do not hesitate to send me your comments on the treatment of a topic or the errors that may have inadvertently crept into this work.

Abstract Algebra



Publisher : PHI Learning ISBN : 9788120339323 Author : I. H. Sheth