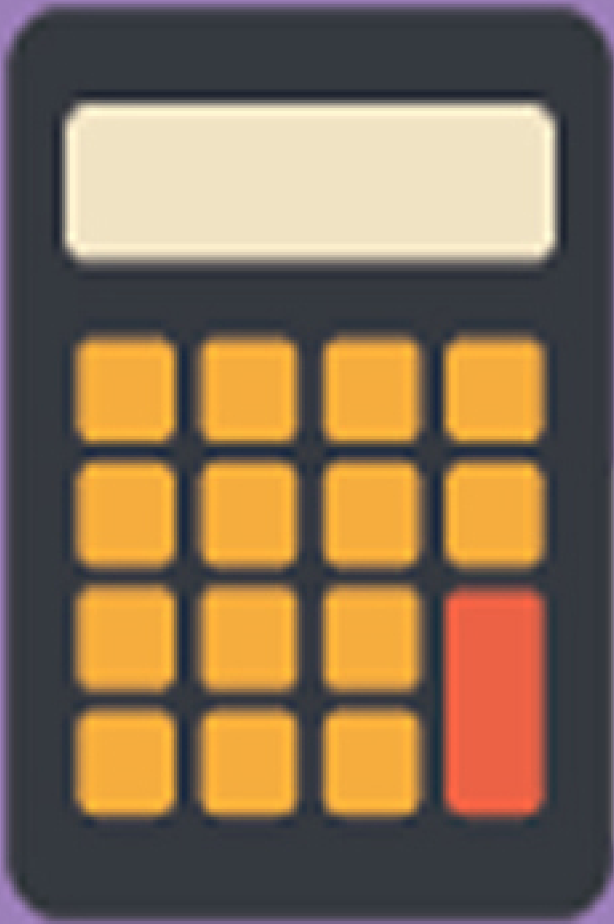
Class-XII



Maths Chapterwise Advanced Study Material

XII Maths Advanced Study Material

Session : 2015-17

1. APPLICATIONS OF MATRICES AND DETERMINANTS

1.1. Introduction :

The students are already familiar with the basic definitions, the elementary operations and some basic properties of matrices. The concept of division is not defined for matrices. In its place and to serve similar purposes, the notion of the inverse of a matrix is introduced. In this section, we are going to study about the inverse of a matrix. To define the inverse of a matrix, we need the concept of adjoint of a matrix.

1.2 Adjoint :

Let $A = [a_{ij}]$ be a square matrix of order n. Let A_{ij} be the cofactor of a_{ij}. Then the nth order matrix $[A_{ij}]^T$ is called the adjoint of A. It is denoted by adj$A$. Thus the adj$A$ is nothing but the transpose of the cofactor matrix $[A_{ij}]$ of A.

Result : If A is a square matrix of order n, then $A\,(\text{adj}A) = |A|\,I_n = (\text{adj}\,A)\,A$, where I_n is the identity matrix of order n.

Proof : Let us prove this result for a square matrix A of order 3.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Then adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\left.\begin{array}{r}\text{The } (i, j)^{\text{th}} \\ \text{element of } A \text{ (adj } A)\end{array}\right\} = a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = \Delta = |A| \text{ if } i = j$$

$$= 0 \text{ if } i \neq j$$

$$\therefore A\,(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I_3$$

Similarly we can prove that $(\text{adj } A)A = |A|\,I_3$

$\therefore\ A\,(\text{adj } A) = |\,A\,|\,I_3 = (\text{adj } A)\,A$

In general we can prove that $A\,(\text{adj } A) = |\,A\,|\,I_n = (\text{adj } A)\,A$.

Example 1.1 : Find the adjoint of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution: The cofactor of a is d, the cofactor of b is $-c$, the cofactor of c is $-b$ and the cofactor of d is a. The matrix formed by the cofactors taken in order is the cofactor matrix of A.

$$\therefore \text{ The cofactor matrix of } A \text{ is } = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Taking transpose of the cofactor matrix, we get the adjoint of A.

$$\therefore \text{ The adjoint of } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1.2 : Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: The cofactors are given by

$$\text{Cofactor of } 1 = A_{11} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 3$$

$$\text{Cofactor of } 1 = A_{12} = -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -9$$

$$\text{Cofactor of } 1 = A_{13} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$\text{Cofactor of } 1 = A_{21} = -\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4$$

$$\text{Cofactor of } 2 = A_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$\text{Cofactor of } -3 = A_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3$$

$$\text{Cofactor of } 2 = A_{31} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

$$\text{Cofactor of } -1 = A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = 4$$

$$\text{Cofactor of } 3 = A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$\text{The Cofactor matrix of } A \text{ is } [A_{ij}] = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$$

$$\therefore\ adj\ \mathrm{A} = (A_{ij})^T = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

Example 1.3 : If $A = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$, verify the result A (adj A) = (adj A) $A = |\,A\,|\ I_2$

Solution: $\mathrm{A} = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$, $|\,A\,| = \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} = 2$

$$\text{adj } A = \begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}$$

$$A\,(\text{adj } A) = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I_2 \qquad \ldots (1)$$

$$(\text{adj } A)\,A = \begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I_2 \qquad \ldots (2)$$

From (1) and (2) we get

$\therefore A$ (adj A) = (adj A) $A = |\,A\,|\ I_2$.

Example 1.4 : If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, verify A (adj A) = (adj A) $A = |\,A\,|\ I_3$

Solution: In example 1.2, we have found

$$\text{adj } A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1(6-3) - 1\,(3+6) + 1(-1-4) = -11$$

$$A\,(\text{adj}\,A) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

$$= -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -11\, I_3 = |A|\, \mathrm{I}_3 \qquad \ldots(1)$$

$$(\text{adj}\,A)\,A = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix}$$

$$= -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -11\, I_3 = |A|\, \mathrm{I}_3 \qquad \ldots(2)$$

From (1) and (2) we get

$$A(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,\mathrm{I}_3$$

1.3 Inverse :

Let A be a square matrix of order n. Then a matrix B, if it exists, such that $AB = BA = I_n$ is called inverse of the matrix A. In this case, we say that A is an invertible matrix. If a matrix A possesses an inverse, then it must be unique. To see this, assume that B and C are two inverses of A, then

$$AB = BA = \mathrm{I}_n \qquad \ldots(1)$$

$$AC = CA = \mathrm{I}_n \qquad \ldots(2)$$

Now $AB = I_n$

$$\Rightarrow \quad C(AB) = CI_n \Rightarrow (CA)B = C \quad (\because \text{ associative property})$$

$$\Rightarrow \quad I_nB = C \Rightarrow B = C$$

i.e., The inverse of a matrix is unique. Next, let us find a formula for computing the inverse of a matrix.

We have already seen that, if A is a square matrix of order n, then

$$A(\text{adj}\,A) = (\text{adj}\,A)A = |A|\,\mathrm{I}_n$$

If we assume that A is non-singular, then $|A| \neq 0$.

Dividing the above equation by $|A|$, we get

$$A\left\{\frac{1}{|A|}(\text{adj } A)\right\} = \left\{\frac{1}{|A|}(\text{adj } A)\right\} A = I_n.$$

From this equation it is clear that the inverse of A is nothing but $\frac{1}{|A|}$ (adj A). We denote this by A^{-1}.

Thus we have the following formula for computing the inverse of a matrix through its adjoint.

If A is a non-singular matrix, there exists an inverse which is given by $A^{-1} = \frac{1}{|A|}$ (adj A).

1.3.1 Properties :

1. Reversal Law for Inverses :

If A, B are any two non-singular matrices of the same order, then AB is also non-singular and

$$(AB)^{-1} = B^{-1} A^{-1}$$

i.e., the inverse of a product is the product of the inverses taken in the reverse order.

Proof : Since A and B are non-singular, $|A| \neq 0$ and $|B| \neq 0$.

We know that $|AB| = |A| \; |B|$

$|A| \neq 0, \quad |B| \neq 0 \;\Rightarrow\; |A| \; |B| \neq 0 \;\Rightarrow\; |AB| \neq 0$

Hence AB is also non-singular. So AB is invertible.

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= AIA^{-1} = AA^{-1} = \mathrm{I}$$

Similarly we can show that $(B^{-1}A^{-1})(AB) = I$

$$\therefore (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

$\therefore$ $B^{-1} A^{-1}$is the inverse of AB.

$$\therefore \; (AB)^{-1} = B^{-1} A^{-1}$$

2. Reversal Law for Transposes (without proof) :

If A and B are matrices conformable to multiplication, then $(AB)^T = B^T A^T$.

i.e., the transpose of the product is the product of the transposes taken in the reverse order.

3. For any non-singular matrix A, $(A^T)^{-1} = (A^{-1})^T$

Proof : We know that $AA^{-1} = I = A^{-1}A$

Taking transpose on both sides of $AA^{-1} = I$, we have $(AA^{-1})^T = I^T$

By reversal law for transposes we get

$$(A^{-1})^T A^T = I \qquad \dots (1)$$

Similarly, by taking transposes on both sides of $A^{-1}A = I$, we have

$$A^T(A^{-1})^T = I \qquad \dots (2)$$

From (1) & (2)

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I$$

$\therefore$ $(A^{-1})^T$ is the inverse of A^T

i.e., $\quad (A^T)^{-1} = (A^{-1})^T$

1.3.2 Computation of Inverses

The following examples illustrate the method of computing the inverses of the given matrices.

Example 1.5 : Find the inverses of the following matrices :

$$\text{(i)} \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Solution:

(i) Let $A = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$, Then $|A| = \begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} = 2 \neq 0$

A is a non-singular matrix. Hence it is invertible. The matrix formed by the cofactors is

$$[A_{ij}] = \begin{bmatrix} -4 & -1 \\ -2 & -1 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ (adj } A) = \frac{1}{2}\begin{bmatrix} -4 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(ii) Let $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. then $|A| = \begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} = 0$

A is singular. Hence A^{-1} does not exist.

(iii) Let $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$. Then $|A| = \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix}$

$$= \cos^2\alpha + \sin^2\alpha = 1 \neq 0$$

$\therefore$ A is non singular and hence it is invertible

$$\text{Adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ (Adj } A) = \frac{1}{1}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

(iv) Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{vmatrix} = 2 \neq 0$

A is non-singular and hence A^{-1} exists

$$\text{Cofactor of } 3 = A_{11} = \begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} = 2$$

$$\text{Cofactor of } 1 = A_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$$

$$\text{Cofactor of } -1 = A_{13} = \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6$$

$$\text{Cofactor of } 2 = A_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$$

$$\text{Cofactor of } -2 = A_{22} = \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\text{Cofactor of } 0 = A_{23} = -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -5$$

$$\text{Cofactor of } 1 = A_{31} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -2$$

$$\text{Cofactor of } 2 = A_{32} = -\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -2$$

$$\text{Cofactor of } -1 = A_{33} = \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = -8$$

$$[A_{ij}] = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}; \text{ adj } A = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{1}{2} \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ 1 & -1 & -1 \\ 3 & -\frac{5}{2} & -4 \end{bmatrix}$$

Example 1.6 : If $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1} A^{-1}$.

Solution:

$|A| = -1 \neq 0$ and $|B| = 1 \neq 0$

So A and B are invertible.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1 \neq 0. \text{ So } AB \text{ is invertible.}$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} (\text{adj } A) = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{ (adj } B) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{adj } AB = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} \text{ (adj } AB) = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \qquad \ldots (1)$$

$$B^{-1} A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \qquad \ldots (2)$$

From (1) and (2) we have $(AB)^{-1} = B^{-1} A^{-1}$.

EXERCISE 1.1

(1) Find the adjoint of the following matrices :

(i) $\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

(2) Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and verify the result

A (adj A) = (adj A)A = $|A| \,.\, I$

(3) Find the adjoint of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and verify the result

A (adj A) = (adj A)A = $|A| \,.\, I$

(4) Find the inverse of each of the following matrices :

(i) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(5) If $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ verify that

(i) $(AB)^{-1} = B^{-1} A^{-1}$ (ii) $(AB)^T = B^T A^T$

(6) Find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ and verify that $A^3 = A^{-1}$

(7) Show that the adjoint of $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ is $3A^T$.

(8) Show that the adjoint of $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ is A itself.

(9) If $A = \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$, prove that $A^{-1} = A^T$.

(10) For $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$, show that $A = A^{-1}$

1.3.3 Solution of a system of linear equations by Matrix Inversion method :

Consider a system of n linear non-homogeneous equations in n unknowns $x_1, x_2, x_3 \ldots\ldots\ldots x_n$.

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots + a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2n} x_n = b_2$

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$a_{n1} x_1 + a_{n2} x_2 + \ldots\ldots\ldots\ldots\ldots + a_{nn} x_n = b_n$

This is of the form $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ an_1 & an_2 & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ \dots \\ b_n \end{bmatrix}$

Thus we get the matrix equation $AX = B$ … (1) where

A = $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix}$; $B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ \dots \\ b_n \end{bmatrix}$

If the coefficients matrix A is non-singular, then A^{-1} exists. Pre-multiply both sides of (1) by A^{-1} we get

$$\mathrm{A}^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B$$

$X = A^{-1}B$ is the solution of (1)

Thus to determine the solution vector X we must compute A^{-1}. Note that this solution is unique.

Example 1.7 : Solve by matrix inversion method $x + y = 3$, $2x + 3y = 8$

Solution:

The given system of equations can be written in the form of

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$AX = B$$

Here $|A| = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \neq 0$

Since A is non-singular, A^{-1} exists.

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

The solution is $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x = 1, \quad y = 2$$

Example 1.8 : Solve by matrix inversion method $2x - y + 3z = 9$, $x + y + z = 6$, $x - y + z = 2$

Solution : The matrix equation is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$A X = B$, where $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = -2 \neq 0$$

A is a non-singular matrix and hence A^{-1} exists.

The cofactors are $A_{11} = 2$, $A_{12} = 0$, $A_{13} = -2$

$A_{21} = -2$, $A_{22} = -1$, $A_{23} = 1$, $A_{31} = -4$, $A_{32} = +1$, $A_{33} = 3$

The matrix formed by the cofactors is

$$[A_{ij}] = \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\text{The adjoint of } A = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \text{adj } A$$

$$\text{Inverse of } A = \frac{1}{|A|}\ (\text{adj } A)$$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

The solution is given by $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore\ x = 1,\ y = 2, z = 3$

EXERCISE 1.2

Solve by matrix inversion method each of the following system of linear equations :

(1) $2x - y = 7$, $3x - 2y = 11$

(2) $7x + 3y = -1$, $2x + y = 0$

(3) $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x + y - z = 0$

(4) $2x - y + z = 7$, $3x + y - 5z = 13$, $x + y + z = 5$

(5) $x - 3y - 8z + 10 = 0$, $3x + y = 4$, $2x + 5y + 6z = 13$

1.4 Rank of a Matrix :

With each matrix, we can associate a non-negative integer, called its rank. The concept of rank plays an important role in solving a system of homogeneous and non-homogeneous equations.

To define rank, we require the notions of submatrix and minor of a matrix. A matrix obtained by leaving some rows and columns from the matrix *A* is called a submatrix of *A*. In particular *A* itself is a submatrix of *A*, because it is obtained from *A* by leaving no rows or columns. The determinant of any square submatrix of the given matrix *A* is called a minor of *A*. If the square submatrix is of order *r*, then the minor is also said to be of order *r*.