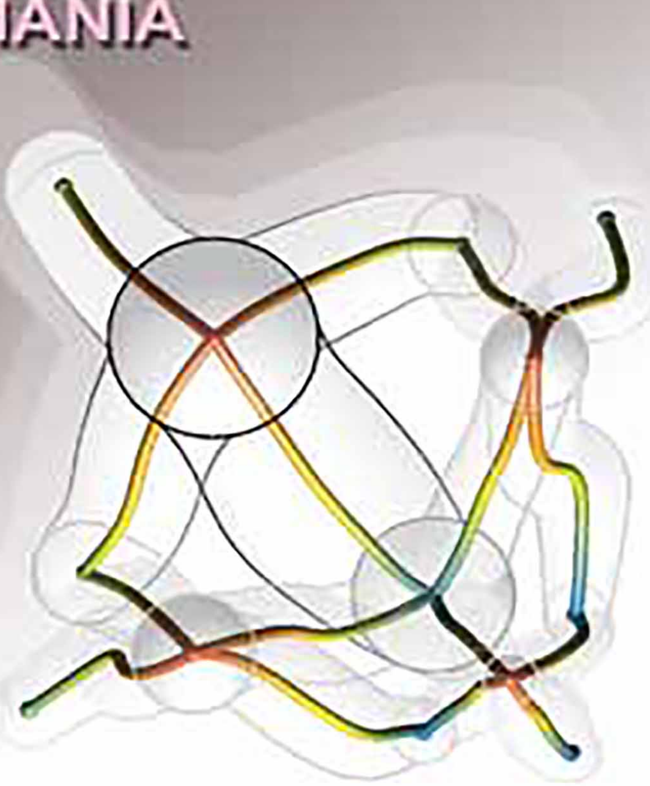
Revised Edition

ELEMENTS OF REAL ANALYSIS

As per UGC Syllabus

SHANTI NARAYAN
Dr. M.D. RAISINGHANIA



S. CHAND

ELEMENTS OF
REAL ANALYSIS

[For B.A., B.Sc. and Honours (Mathematics and Physics), M.A. and M.Sc. (Mathematics) students of various Universities/Institutions as per UGC Model Carriculum. Also useful for GATE and various other competitive examinations]

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PREFACE TO THE FOURTEENTH EDITION

References to the latest papers of various universities and GATE have been inserted at proper places. More additional problems have been inserted in almost each chapter of this book. New topics have been inserted in some chapters. I hope that these changes will make the material of this book more useful to the reader.

All valuable suggestions for further improvement of the book will be highly appreciated.

M.D. Raisinghania

PREFACE TO THE EIGHTH EDITION

The book originally written, about 40 years ago, has during the intevening period, been revised and reprinted several times. Due to the new U.G.C. Model syllabus and the demand for more matter from the students and teachers, a thorough revision of the book was overdue. I very humbly took the challenge of revising this perfect well-written book of late Shri Shanti Narayan with whom I had personal contact from 1962 onwards. He was my teacher and guiding star in art of writting a book.

I have tried to meet the rapidly changing demands of students interested in self-study and appearing in various examinations. Accordingly, a large variety of illustrative solved examples have been included in every chapter. References to the latest papers of various universities and I.A.S. examination have been inserted at proper places.

The following new chapters have been added in this present edition

• Countability of sets • The Riemann-Stieltjes Integrals • Uniform convergence of sequences and series of functions • Improper Integrals • Metric spaces

The book, in the present form, is a humble effort to make it more useful to the students and teachers.

I am extremely thankfull to the Managing Director, Shri Ravindra Kumar Gupta, Shri Navin Joshi, Vice President (Publishing) and Advisor, Shri R.S. Saxena for personal intersest throughout the preparation of the book. My sincere thanks are also due to Mr. Shishir Bhatnagar of S. Chand & Company for bringing this book in an excellent form.

All valuable suggestions for further improvement of the book will be highly appreciated

M.D. Raisinghania

Preface to the First Edition

This book is an attempt to present Elements of Real Analysis to under-graduate students,* on the basis of the University Grants Commission Review Committee report recommendations and several Universities having provided a course along the lines of these recomendations. This book must not, however, be thought of an abridged edition of the Author's "A Course of Mathematical Analysis" for M.A. students.

Chapter I provides description of Set of Real Number as a complete ordered field and no attempt has been made to construct the Set starting from some Axioms. Chapter II deals with bounds and limit points of sets of real numbers. Chapter III concerns itself with Real sequences defined as functions on the set of Natural numbers into the set of Real numbers. This is followed by Chapter IV of Infinite Series dealing with mostly convergence tests for positive term series as also a test on alternating series. Chapter V deals with the nature of the range of a real valued continuous function with a closed finite interval as its domain. Chapter VI on Derivability deals with the rigorous proof of Role's theorem as also of Lagrange's theorem. Chapter VIII deals with Riemann Integrability.

The book contains some examples and exercises meant only to help a proper undertaking of the text. The book also seeks to adopt comparatively more modern notation for the subject.

It is hoped that this "Elements of Real Analysis" will provide a stimulus for the commencement of the study of Analysis at undergraduate stage which has already been so much delayed.

March, 1965 **SHANTI NARAYAN**

CONTENTS

Dedicated to memory

of my parents

— M.D. Raisinghania

SYMBOLS

$\boldsymbol{N}$	the set of natural numbers
$\boldsymbol{Z}$ **or** $\boldsymbol{I}$	the set of integers
$\boldsymbol{Z}^+$ **or** $\boldsymbol{I}^+$	the set of positive integers
$\boldsymbol{Q}$	the set of rational numbers
$\boldsymbol{Q}^+$	the set of positive rational numbers
$\boldsymbol{R}$	the set of real numbers
$\boldsymbol{R}^+$	the set of positive real numbers
$\Rightarrow$	implies
$\Leftrightarrow$	is equivalent to
$\{\ \}$	set
$\in$	is an element of
$:$	such that
$\subset$	is contained in (is a subset of)
$\supset$	contains (is a superset of)
A' or $\sim A$ or A^c or $U \sim A$ or $U \setminus A$	complement of A with respect to U
A' or $\sim A$ or A^c or $\mathbf{R} \sim A$ or $\mathbf{R} \setminus A$	complement of A with respect to $\boldsymbol{R}$
$\cup$	union
$\cap$	intersection
ϕ	the empty set
$\exists$	there exists
$\forall$	for all

THE GREEK ALPHABET

alpha	α	A
beta	β	B
gamma	γ	Γ
delta	δ	Δ
epsilon	ε	E
zeta	ζ	Z
eta	η	H
theta	θ	Θ
iota	ι	I
kappa	κ	K
lambda	λ	Λ
mu	μ	M
nu	ν	N
xi	ξ	Ξ
omicron	o	O
pi	π	Π
rho	ρ	P
sigma	σ	Σ
tau	τ	T
upsilon	υ	Υ
phi	ϕ	Φ
chi	χ	X
psi	ψ	Ψ
omega	ω	Ω

Chapter

1

Sets and Functions

1.1. INTRODUCTION

Set-Theoretic Notation and Terminology. What is called *Real Analysis* is a development of the set of real numbers which is reached through a series of successive extensions and generalisations starting from the set of natural numbers. As a matter of fact, starting from the set of natural numbers, we first pass on to the set of integers, then to the set of rational numbers and finally to the set of real numbers. Of course, from the set of real numbers, we can also pass on to the set of complex numbers but since Real Analysis is not concerned with complex numbers, we shall have nothing to do with complex numbers in this course. What is known as *Complex Analysis* is a development of the set of complex numbers.

It is no part of the plan of this book to *construct* the various systems of numbers and to develop their properties axiomatically from a given system of postulates. All that is intended here is to *describe* the various properties of these systems and to bring out the essential differences between them; it being understood that most of these properties are already familiar to the reader. What is important from the point of view of this course is the form in which these properties are stated and the type of emphasis which is thus brought out. While this programme will be undertaken in the following Chapter 2, we propose in this chapter to introduce some new notations and terms pertaining to '*sets*' and '*functions*' which will be found useful for the exposition of the subject proposed to be studied in this book.

1.2. STATEMENTS

In our everyday language, we are concerned with statements which are often distinguished as Interrogative, Imperative, Exclamatory or Declarative. In Mathematics, however, our chief interest is only in those statements which are Declarative and which may be either true or false. Consider, for example, the following statements, some of which are true and some false :

(*i*) The sum of the three angles of a triangle is equal to two right angles.

(*ii*) Every rectangle is a triangle.

(*iii*) The sum of an opposite pair of angles of a cyclic quadrilateral is equal to two right angles.

(*iv*) If two straight lines are perpendicular to the same straight line, then they are parallel to each other.

(*v*) The straight line joining the mid-points of two sides of a triangle is parallel to the third.

(*vi*) If x is 2 and y is 5, then $x + y$ is 7.

(*vii*) If $xy = 0$ and x, y are real numbers, then $x = 0$ and $y = 0$.

(*viii*) If $xy = 1$ and x, y are natural numbers, then $x = 1$ and $y = 1$.

(*ix*) If $xy = 1$ and x, y are rational numbers, then $x = 1$ and $y = 1$.

(*x*) If $x = 3$, then $x^2 = 9$.

(*xi*) If $x^2 = 9$, then $x = 3$.

(*xii*) Every pair of natural numbers admits of a highest common factor.

(*xiii*) Every natural number admits of an infinite number of factors.

1.3. CONNECTIVES : $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$, ~

In the study of Mathematics, we are concerned with logical inter-connections between statements. Also on the basis of given statements, we build up new statements. There are a few symbols which are found useful for describing logical inter-connections between given statements and for building up new statements from the old ones and we now proceed to introduce these symbols.

The Connective $\Rightarrow$: If P and Q be two statements such that the truth of the statement P implies the truth of the statement Q, we exhibit this relationship between the two statements symbolically as $P \Rightarrow Q$ so that the symbol, $\Rightarrow$, stands for '*Implies*'. Thus, we have

(*i*) $x = 3 \Rightarrow x^2 = 9$. (*ii*) *x is a real number* $\Rightarrow x^2 \geq 0$.

(*iii*) *ABCD is a parallelogram* $\Rightarrow AB = CD$. (*iv*) *ABC is a triangle* $\Rightarrow AB + BC > AC$.

(*v*) *AB* □ *CD and CD* □ *EF* $\Rightarrow$ *AB* □ *EF*.

The statement $P \Rightarrow Q$ formulates what is often expressed in any one of the following ways :

(*i*) If P then Q.

(*ii*) A necessary condition for the truth of P is the truth of Q.

(*iii*) A sufficient condition for the truth of Q is the truth of P.

The symbol $\Rightarrow$ is a *connective* in as much as connecting the two statements P, Q, it generates a third statement, *viz.*, the statement $P \Rightarrow Q$.

The Connective $\Leftrightarrow$: If P, Q are two statements such that we have $P \Rightarrow Q$ as well as $Q \Rightarrow P$, we write $P \Leftrightarrow Q$ and say that P implies and is implied by Q or that P is *equivalent* to Q.

Thus, *ABCD is a parallelogram* $\Leftrightarrow AB = CD$ *and* $BC = AD$.

$x^2 = 9$, x, y are real numbers $\Leftrightarrow x \in \{3, -3\}$.

The statement $P \Leftrightarrow Q$ expresses what is also described in one of the following ways :

(*i*) P if and only if Q. (*ii*) Q if and only if P.

(*iii*) A necessary and sufficient condition for the truth of P is the truth of Q.

(*iv*) A necessary and sufficient condition for the truth of Q is the truth of P.

(*v*) P and Q are equivalent statements.

The Connectives $\wedge$, $\vee$: The symbols $\wedge$, $\vee$, stand for *and*, *or* respectively.

If P and Q be two statements, then with the help of these connectives, we form two new statements $P \wedge Q$, $P \vee Q$.

The statement $P \wedge Q$ is true if and only if P is true *as well as* Q is true. Thus, if the statements P, Q are *both* true, then the statement $P \wedge Q$ is true and *vice-versa*.

The statement $P \vee Q$ is true if and only if P is true or Q is true, *i.e.*, if and only if at least one of P and Q is true.

For example, *ABCD* is a parallelogram $\Leftrightarrow AB = CD \wedge BC = AD$.

$x^2 - 5x + 6 = 0 \Rightarrow x = 2 \vee x = 3$.

$xy = 0$, x, y are real numbers $\Leftrightarrow x = 0 \vee y = 0$.

$x = 2 \wedge y = 3 \Rightarrow x + y = 5$.

Negation ~ : If P denotes a statement, then $\sim P$, denotes the *negation* or the *denial* of P.

Let P denote the statement $x = 4$. Then, $\sim P$, denotes the statement $x \neq 4$.

For example, *ABC* is a triangle $\Rightarrow \sim [AB + BC \leq AC]$.

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