INTRODUCTION TO FORCE AND DISPLACEMENT METHODS OF STRUCTURAL ANALYSIS

Since twentieth century, indeterminate structures are being widely used for its obvious merits. It may be recalled that, in the case of indeterminate structures either the reactions or the internal forces cannot be determined from equations of statics alone. In such structures, the number of reactions or the number of internal forces exceeds the number of static equilibrium equations. In addition to equilibrium equations, compatibility equations are used to evaluate the unknown reactions and internal forces in statically indeterminate structure. In the analysis of indeterminate structure it is necessary to satisfy the equilibrium equations (implying that the structure is in equilibrium) compatibility equations (requirement if for assuring the continuity of the structure without any breaks) and force displacement equations (the way in which displacement are related to forces). We have two distinct method of analysis for statically indeterminate structure depending upon how the above equations are satisfied:

1. Force method of analysis

2. Displacement method of analysis

In the force method of analysis, primary unknown are forces. In this method compatibility equations are written for displacement and rotations (which are calculated by force displacement equations). Solving these equations, redundant forces are calculated. Once the redundant forces are calculated, the remaining reactions are evaluated by equations of equilibrium.

In the displacement method of analysis, the primary unknowns are the displacements. In this method, first force -displacement relations are computed and subsequently equations are written satisfying the equilibrium conditions of the structure. After determining the unknown displacements, the other forces are calculated satisfying the compatibility conditions and force displacement relations. The displacement-based method is amenable to computer programming and hence the method is being widely used in the modern day structural analysis.

DIFFERENCE BETWEEN FORCE & DISPLACEMENT METHODS

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Types of indeterminacy: static indeterminacy

Types of indeterminacy: kinematic indeterminacy
All displacement methods follow the above general procedure. The Slope-deflection and moment distribution methods were extensively used for many years before the computer era. In the displacement method of analysis, primary unknowns are joint displacements which are commonly referred to as the degrees of freedom of the structure. It is necessary to consider all the independent degrees of freedom while writing the equilibrium equations. These degrees of freedom are specified at supports, joints and at the free ends.

**SLOPE DEFLECTION METHOD**

In the slope-deflection method, the relationship is established between moments at the ends of the members and the corresponding rotations and displacements.

The slope-deflection method can be used to analyze statically determinate and indeterminate beams and frames. In this method it is assumed that all deformations are due to bending only. In other words deformations due to axial forces are neglected. In the force method of analysis compatibility equations are written in terms of unknown reactions. It must be noted that all the unknown reactions appear in each of the compatibility equations making it difficult to solve resulting equations. The slope-deflection equations are not that lengthy in comparison. The basic idea of the slope deflection method is to write the equilibrium equations for each node in terms of the deflections and rotations. Solve for the generalized displacements. Using moment-displacement relations, moments are then known. The structure is thus reduced to a determinate structure. The slope-deflection method was originally developed by Heinrich Manderla and Otto Mohr for computing secondary stresses in trusses. The method as used today was presented by G.A.Maney in 1915 for analyzing rigid jointed structures.

Fundamental Slope-Deflection Equations:

The slope deflection method is so named as it relates the unknown slopes and deflections to the applied load on a structure. In order to develop general form of slope deflection equations, we will consider the typical span AB of a continuous beam which is subjected to arbitrary loading and has a constant EI. We wish to relate the beams internal end moments $M_{AB} & M_{BA}$ in terms of its three degrees of freedom, namely its angular displacements $\Theta_A & \Theta_B$ and linear displacement $\Delta$ which could be caused by relative settlements between the supports. Since we will be developing a formula, moments and angular displacements will be considered positive, when they act clockwise on the span. The linear displacement $\Delta$ will be considered positive since this displacement causes the chord of the span and the span’s chord angle to rotate clockwise. The slope deflection equations can be obtained by
using principle of superposition by considering separately the moments developed at each supports due to each of the displacements $\theta_A$ & $\theta_B$ & $\Delta$ and then the loads.

Case A: fixed-end moments

$$M_{AB} = FEM_{AB}, \quad M_{BA} = FEM_{BA}$$

$$FEM_{AB} = -\frac{wl^2}{12}, \quad FEM_{BA} = \frac{wl^2}{12}$$
Consider node A of the member as shown in figure to rotate $\theta_A$ while its far end B is fixed. To determine the moment $M_{AB}$ needed to cause the displacement, we will use conjugate beam method. The end shear at A acts downwards on the beam since $\theta_A$ is clockwise.

\[ \Sigma M_A = 0, \quad \left[ \frac{1}{2} \frac{M_{AB}}{EI} L \right] \frac{L}{3} - \left[ \frac{1}{2} \frac{M_{BA}}{EI} L \right] \frac{2L}{3} = 0 \]

\[ \Sigma M_B = 0, \quad \left[ \frac{1}{2} \frac{M_{BA}}{EI} L \right] \frac{L}{3} - \left[ \frac{1}{2} \frac{M_{BA}}{EI} L \right] \frac{2L}{3} + \theta_A L = 0 \]

\[ M_{AB} = \frac{4EI}{L} \theta_A, \quad M_{BA} = \frac{2EI}{L} \theta_A \]

Case C: rotation at B, $\theta_B$ (angular displacement at B)

In a similar manner if the end B of the beam rotates to its final position, $\theta_B$ while end A is held fixed. We can relate the applied moment $M_{BA}$ to the angular displacement $\theta_B$ and the reaction moment $M_{AB}$. 

\[ \Sigma M_A = 0, \quad \left[ \frac{1}{2} \frac{M_{BA}}{EI} L \right] \frac{L}{3} - \left[ \frac{1}{2} \frac{M_{BA}}{EI} L \right] \frac{2L}{3} = 0 \]

\[ \Sigma M_B = 0, \quad \left[ \frac{1}{2} \frac{M_{AB}}{EI} L \right] \frac{L}{3} - \left[ \frac{1}{2} \frac{M_{AB}}{EI} L \right] \frac{2L}{3} + \theta_B L = 0 \]

\[ M_{BA} = \frac{4EI}{L} \theta_B, \quad M_{AB} = \frac{2EI}{L} \theta_B \]
\( M_{AB} = \frac{2EI}{L} \theta_B, \quad M_{BA} = \frac{4EI}{L} \theta_B \)

Case D: displacement of end B related to end A

If the far node B of the member is displaced relative to A so that the chord of the member rotates clockwise (positive displacement). The moment \( M \) can be related to displacement \( \Delta \) by using conjugate beam method. The conjugate beam is free at both the ends as the real beam is fixed supported. Due to displacement of the real beam at B, the moment at the end B of the conjugate beam must have a magnitude of \( \Delta \). Summing moments about B we have,

\[
\Sigma M_B = 0, \quad \left[ \frac{1}{2EI} \right] \int_0^L \left( \frac{1}{2EI} \right) (L) \frac{2L}{3} - \left[ \frac{1}{2EI} \right] \int_0^L (L) \frac{L}{3} - \Delta = 0
\]

\[
M_{AB} = M_{BA} = M = -\frac{6EI}{L^2} \Delta
\]

By our sign convention the induced moment is negative, since for equilibrium it acts counter clockwise on the member.

If the end moments due to the loadings and each displacements are added together, then the resultant moments at the ends can be written as,
\[ M_{AB} = \frac{2EI}{L} \left[ 2\theta_A + \theta_B - \frac{3\Delta}{L} \right] + \text{FEM}_{AB} \]

\[ M_{BA} = \frac{2EI}{L} \left[ 2\theta_B + \theta_A - \frac{3\Delta}{L} \right] + \text{FEM}_{BA} \]

**Fixed end moment table**

<table>
<thead>
<tr>
<th>Configuration</th>
<th>( (\text{FEM})_{AB} )</th>
<th>( (\text{FEM})_{BA} )</th>
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<tbody>
<tr>
<td></td>
<td>( \frac{PL}{8} )</td>
<td>( \frac{PL}{8} )</td>
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<tr>
<td></td>
<td>( \frac{Pl^2a}{L^2} )</td>
<td>( \frac{Pb^2}{2L^2} )</td>
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<tr>
<td></td>
<td>( \frac{2PL}{9} )</td>
<td>( \frac{2PL}{9} )</td>
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<tr>
<td></td>
<td>( \frac{5PL}{36} )</td>
<td>( \frac{5PL}{16} )</td>
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<td>( \frac{3PL}{16} )</td>
<td>( \frac{PL}{16} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{45PL}{96} )</td>
<td>( \frac{45PL}{96} )</td>
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General Procedure OF Slope-Deflection Method

- Find the fixed end moments of each span (both ends left & right).
- Apply the slope deflection equation on each span & identify the unknowns.
- Write down the joint equilibrium equations.
- Solve the equilibrium equations to get the unknown rotation & deflections.
- Determine the end moments and then treat each span as simply supported beam subjected to given load & end moments so we can work out the reactions & draw the bending moment & shear force diagram.
Numerical Examples

1. Q. Analyze two span continuous beam ABC by slope deflection method. Then draw Bending moment & Shear force diagram. Take EI constant.

Fixed end moments are

\[ M_{AB} = -\frac{W_{ab}^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44 \text{ KNm} \]
\[ M_{BA} = \frac{Wa^2b}{L^2} = \frac{100 \times 4^2 \times 2}{6^2} = 88.89 \text{ KNm} \]
\[ M_{BC} = -\frac{wL^2}{12} = -\frac{20 \times 5^2}{12} = -41.67 \text{ KNm} \]
\[ M_{CB} = \frac{wL^2}{12} = \frac{20 \times 5^2}{12} = 41.67 \text{ KNm} \]

Since A is fixed \( \theta_A = 0 \) & \( \theta_B \) & \( \theta_C \neq 0 \)

Slope deflection equations are

\[ M_{AB} = M_{FAB} + \frac{2EI}{L} [2\theta_A + \theta_B] = -44.44 + \frac{2EI}{6} \theta_B = -44.44 + \frac{EI}{3} \theta_B \quad \ldots \quad (1) \]
\[ M_{BA} = M_{FBA} + \frac{2EI}{L} [2\theta_B + \theta_A] = 88.89 + \frac{4EI}{6} \theta_B = 88.89 + \frac{2EI}{3} \theta_B \quad \ldots \quad (2) \]
\[ M_{BC} = M_{FBC} + \frac{2EI}{L} [2\theta_B + \theta_C] = -41.67 + \frac{2EI}{5} \theta_B + \frac{2EI}{5} \theta_C \quad \ldots \quad (3) \]
\[ M_{CB} = M_{FCB} + \frac{2EI}{L} [2\theta_C + \theta_B] = 41.67 + \frac{4EI}{5} \theta_C + \frac{2EI}{5} \theta_B \quad \ldots \quad (4) \]

In all the above 4 equations there are only 2 unknowns \( \theta_B \) & \( \theta_C \) and accordingly the boundary conditions are

\[ M_{BA} + M_{BC} = 0 \]
\[ M_{CB} = 0 \text{ as end C is simply supported.} \]

\[ M_{BA} + M_{BC} = 88.89 + \frac{2EI}{3} \theta_B - 41.67 + \frac{4EI}{5} \theta_B + \frac{2EI}{5} \theta_C = 47.22 + \frac{22}{15} EI\theta_B + \frac{2}{5} EI\theta_C = 0 \quad \ldots \quad (5) \]
\[ M_{CB} = 41.67 + \frac{4EI}{5} \theta_C + \frac{2EI}{5} \theta_B = 0 \quad \ldots \quad (6) \]

Solving the equations (5) & (6), we get
\[ \theta_B = -\frac{20.83}{EI} \]
\[ \theta_C = -\frac{41.67}{EI} \]

Substituting the values in the slope deflections we have,

\[ M_{AB} = -44.44 + \frac{E_1}{3} \left( -\frac{20.83}{EI} \right) = -51.38 \text{ KNm} \]
\[ M_{BA} = 88.89 + \frac{2E_1}{3} \left( -\frac{20.83}{EI} \right) = 75 \text{ KNm} \]
\[ M_{BC} = -41.67 + \frac{4E_1}{5} \left( -\frac{20.83}{EI} \right) + \frac{2E_1}{5} \left( -\frac{41.67}{EI} \right) = -75 \text{ KNm} \]
\[ M_{CB} = 41.67 + \frac{4E_1}{5} \left( -\frac{41.67}{EI} \right) + \frac{2E_1}{5} \left( -\frac{20.83}{EI} \right) = 0 \]

Reactions: Consider the free body diagram of the beam

Find reactions using equations of equilibrium.

Span AB: \( \Sigma M_A = 0 \), \( R_B \times 6 = 100 \times 4 + 75 - 51.38 \)
\[ \therefore R_B = 70.60 \text{ KN} \]
\[ \Sigma V = 0, \quad R_A + R_B = 100 \text{ KN} \]
\[ \therefore R_A = 100 - 70.60 = 29.40 \text{ KN} \]

Span BC: \( \Sigma M_C = 0 \), \( R_B \times 5 = 20 \times 5 \times \frac{5}{2} + 75 \)
\[ \therefore R_B = 65 \text{ KN} \]
\[ \Sigma V = 0 \quad R_B + R_C = 20 \times 5 = 100 \text{ KN} \]
\[ R_C = 100 - 65 = 35 \text{ KN} \]

Using these data BM and SF diagram can be drawn
Max BM:

**Span AB:** Max BM in span AB occurs under point load and can be found geometrically, 
\[ M_{\text{max}} = 113.33 - 51.38 \left( \frac{75 - 51.38}{6} \right) \times 4 = 46.20 \text{KNm} \]

**Span BC:** Max BM in span BC occurs where shear force is zero or changes its sign. Hence consider SF equation w.r.t C
\[ S_x = 35 - 20x = 0 \]
\[ x = \frac{35}{20} = 1.75 \text{m} \]
Max BM occurs at 1.75m from C
\[ \therefore M_{\text{max}} = 35 \times 1.75 - 20 \times \frac{1.75^2}{2} = 30.625 \text{ KNm} \]
2. Q. Analyze continuous beam ABCD by slope deflection method and then draw bending moment diagram. Take EI constant.

\[ \theta_A = 0 \& \theta_B \& \theta_C \neq 0 \]

\[ M_{AB} = -\frac{W_{ab}^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44 \text{ KNm} \]

\[ M_{BA} = \frac{W_{ba}b}{L^2} = \frac{100 \times 4^2 \times 2}{6^2} = 88.89 \text{ KNm} \]

\[ M_{BC} = -\frac{wL^2}{12} = -\frac{20 \times 5^2}{12} = -41.67 \text{ KNm} \]

\[ M_{CB} = \frac{wL^2}{12} = \frac{20 \times 5^2}{12} = 41.67 \text{ KNm} \]

\[ M_{CD} = -20 \times 1.5 = -30 \text{ KNm} \]

Slope deflection equations are

\[ M_{AB} = M_{FAB} + \frac{2EI}{L} [2\theta_A + \theta_B] = -44.44 + \frac{2EI}{6} \theta_B = -44.44 + \frac{EI}{3} \theta_B \quad \cdots (1) \]

\[ M_{BA} = M_{FBA} + \frac{2EI}{L} [2\theta_B + \theta_A] = 88.89 + \frac{4EI}{6} \theta_B = 88.89 + \frac{2EI}{3} \theta_B \quad \cdots (2) \]

\[ M_{BC} = M_{FBC} + \frac{2EI}{L} [2\theta_B + \theta_C] = -41.67 + \frac{4EI}{5} \theta_B + \frac{2EI}{5} \theta_C \quad \cdots (3) \]

\[ M_{CB} = M_{FCB} + \frac{2EI}{L} [2\theta_C + \theta_B] = 41.67 + \frac{4EI}{5} \theta_C + \frac{2EI}{5} \theta_B \quad \cdots (4) \]

\[ M_{CD} = -20 \times 1.5 = -30 \text{ KNm} \]

In all the above equations there are only 2 unknowns \( \theta_B \& \theta_C \) and accordingly the boundary conditions are

\[ M_{BA} + M_{BC} = 0 \]

\[ M_{CB} + M_{CD} = 0 \]

\[ M_{BA} + M_{BC} = 88.89 + \frac{2EI}{3} \theta_B - 41.67 + \frac{4EI}{5} \theta_B + \frac{2EI}{5} \theta_C = 47.22 + \frac{22}{15} EI \theta_B + \frac{2}{5} EI \theta_C = 0 \quad \cdots (5) \]

\[ M_{CB} + M_{CD} = 41.67 + \frac{4EI}{5} \theta_C + \frac{2EI}{5} \theta_B - 30 = 11.67 + \frac{2EI}{5} \theta_B + \frac{4EI}{5} \theta_C = 0 \quad \cdots (6) \]

Solving equations (5) & (6),

\[ \theta_B = -\frac{32.67}{EI} \]

\[ \theta_C = \frac{1.75}{EI} \]
Substituting the values in the slope deflections we have,

\[ M_{AB} = -44.44 + \frac{Ei}{3} \times \left( -\frac{32.67}{Ei} \right) = -61 \text{ KNm} \]

\[ M_{BA} = 88.89 + \frac{2Ei}{3} \times \left( -\frac{32.67}{Ei} \right) = 67.11 \text{ KNm} \]

\[ M_{BC} = -41.67 + \frac{4Ei}{5} \left( -\frac{32.67}{Ei} \right) + \frac{2Ei}{5} \left( \frac{1.75}{Ei} \right) = -67.11 \text{ KNm} \]

\[ M_{CB} = 41.67 + \frac{4Ei}{5} \left( \frac{1.75}{Ei} \right) + \frac{2Ei}{5} \left( -\frac{32.67}{Ei} \right) = 30 \text{ KNm} \]

\[ M_{CD} = -30 \text{ KNm} \]

**Reactions:** Consider free body diagram of beam AB, BC and CD as shown
Span AB:
\[ R_B \times 6 = 100 \times 4 + 67.11 - 61 \]
\[ R_B = 67.69 \text{KN} \]
\[ R_A = 100 - R_B = 32.31 \text{KN} \]

Span BC:
\[ R_C \times 5 = 20 \times \frac{5}{2} \times 5 + 30 - 67.11 \]
\[ R_C = 42.58 \text{KN} \]
\[ R_B = 20 \times 5 - R_C = 57.42 \text{KN} \]

Maximum Bending Moments:
Span AB: Occurs under point load
\[ M_{\text{max}} = 133.33 - 61 - \frac{67.11 - 61}{6} \times 4 = 68.26 \text{KNm} \]
Span BC: Where SF=0, consider SF equation with C as reference
\[ S_x = 42.58 - 20x = 0 \]
\[ x = \frac{42.58}{20} = 2.13 \text{m} \]
\[ M_{\text{max}} = 42.58 \times 2.13 - 20 \times \frac{2.13^2}{2} = 15.26 \text{KNm} \]

3. Q. Analyse the continuous beam ABCD shown in figure by slope deflection method. The support B sinks by 15mm. Take \( E = 200 \times 10^6 \text{KN/m}^2 \) and \( I = 120 \times 10^6 \text{m}^4 \)
A. $\Theta_A = 0$ & $\Theta_B$ & $\Theta_C \neq 0$ $\Delta = 15mm$

\[
M_{AB} = -\frac{Wab^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44 \text{ KNm}
\]

\[
M_{BA} = \frac{Wa^2b}{L^2} = \frac{100 \times 4^2 \times 2}{6^2} = 88.89 \text{ KNm}
\]

\[
M_{BC} = -\frac{wl^2}{12} = -\frac{20 \times 5^2}{12} = -41.67 \text{ KNm}
\]

\[
M_{CB} = \frac{wl^2}{12} = \frac{20 \times 5^2}{12} = 41.67 \text{ KNm}
\]

\[
M_{CD} = -20 \times 1.5 = -30 \text{ KNm}
\]

FEM due to yield of support B

For span AB:

\[
M_{AB} = M_{BA} = -\frac{6EI}{L^2} \Delta = -\frac{6 \times 200 \times 10^5 \times 120 \times 10^{-6}}{6^2} \frac{15}{1000} = -6 \text{ KNm}
\]

For span BC:

\[
M_{BC} = M_{CB} = \frac{6EI}{L^2} \Delta = \frac{6 \times 200 \times 10^5 \times 120 \times 10^{-6}}{5^2} \frac{15}{1000} = 8.64 \text{ KNm}
\]

Slope deflection equations are

\[
M_{AB} = M_{FAB} + \frac{2EI}{6} \left[ 2\theta_A + \theta_B - \frac{3\Delta}{6} \right] = -44.44 + \frac{EI}{3} \theta_B - 6 = -50.44 + \frac{EI}{3} \theta_B \quad \ldots \quad (1)
\]

\[
M_{BA} = M_{FBA} + \frac{2EI}{6} \left[ 2\theta_B + \theta_A - \frac{3\Delta}{6} \right] = 88.89 + \frac{2EI}{3} \theta_B - 6 = 82.89 + \frac{2EI}{3} \theta_B \quad \ldots \quad (2)
\]

\[
M_{BC} = M_{FBC} + \frac{2EI}{5} \left[ 2\theta_B + \theta_C + \frac{3\Delta}{5} \right] = -41.67 + \frac{4EI}{5} \theta_B + \frac{2EI}{5} \theta_C + 8.64
\]

\[
= -33.03 + \frac{4EI}{5} \theta_B + \frac{2EI}{5} \theta_C \quad \ldots \quad (3)
\]

\[
M_{CB} = M_{FCB} + \frac{2EI}{5} \left[ 2\theta_C + \theta_B + \frac{3\Delta}{5} \right] = 41.67 + \frac{4EI}{5} \theta_C + \frac{2EI}{5} \theta_B + 8.64
\]

\[
= 50.31 + \frac{4EI}{5} \theta_C + \frac{2EI}{5} \theta_B \quad \ldots \quad (4)
\]

\[
M_{CD} = -20 \times 1.5 = -30 \text{ KNm}
\]

In all the above equations there are only 2 unknowns $\theta_B$ & $\theta_C$ and accordingly the boundary conditions are

\[
M_{BA} + M_{BC} = 0
\]

\[
M_{CB} + M_{CD} = 0
\]
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