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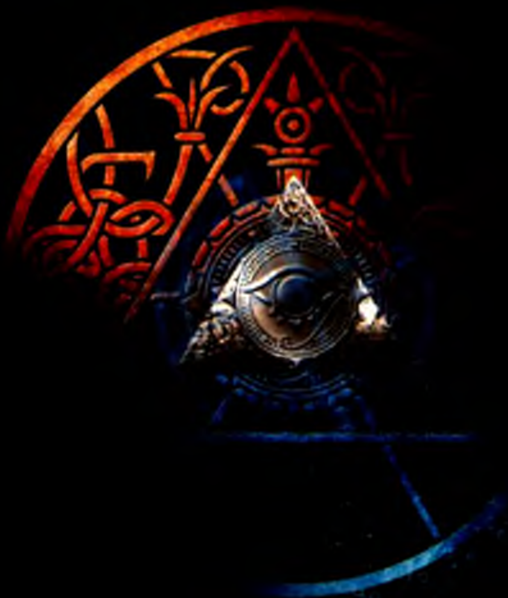

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BCA MATHEMATICS—103

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(Strictly as per M.D.U., Rohtak Syllabus)

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Syllabus

MATHEMATICS-(BCA—103)
(For BCA Students of M.D.U., Rohtak)

Matrices and Determinants, Permutation and Combination, Binomial Theorem with Examples.

Concepts of Countability of Sets—Supremum and Infimum of a Set, Neighbourhood of a Point, Interior and Limit Point of a Set.

Sequences—Sequence, Convergent Sequences, Divergent Sequences, Cauchy Sequence, Monotonic Sequence, Subsequence, Limit Superior and Limit Inferior of Sequence.

Infinite Series—Convergence of Series, Positive Term Series, Comparison Tests, D' Alembert's Ratio Test, Cauchy's *n*th Root Test, Raabe's Test, Gauss's Test, Cauchy's Integral Test, Alternating Series, Absolute and Conditional Convergence.

Taylor's Series and Maclaurin's Series, Applications of Mean Value Theorem to Monotone Functions and Inequalities, Indeterminate Forms.

Preface

The present edition of the book is strictly according to the syllabus of BCA Mathematics paper 103 of M.D. University, Rohtak. The book is self-enplanatory. Minimum theoretical part is incorporated to understand the problems. The authors in the text desire to make the students fear free of maths. The motivation to write the book is feedback from the students to provide them upto the mark and relevent material for the required course.

The authors will feel highly oblized if suggestions in the direction to improve the text are conveyed.

—Authors

1

Algebra of Matrices

1.1. INTRODUCTION

In this chapter we shall describe matrices, some special type of arrangement of numbers. The use of matrices helps a lot in mathematical investigations and has become an integral part of mathematics. Contribution of matrices in the fields of business, physics, engineering etc., is indispensable.

1.1.1. Definition of a Matrix. A rectangular arrangement of numbers in a finite number of rows and columns, enclosed in a pair of brackets '[ ]' or '( )' is called a matrix *e.g.*, $A = \begin{bmatrix} 1 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$ is a rectangular arrangement of numbers which has two rows (horizontal lines) '1 5 6', '3 2 1' and three columns (vertical lines) $\begin{matrix} 1' & 5' & 6 \\ 3 & 2 & 1' \end{matrix}$

Observe that the element 5 lies in Ist row and second column. Similarly, we can tell the position of every entry or element of the matrix A.

1.1.2. Order of a Matrix. If a matrix has 'm' rows and 'n' columns, then the order of the matrix is '$m \times n$' (read as m by n), *e.g.*, the order of the matrix 'A' above is 2×3.

Note. A matrix of order '$m \times n$' has 'mn' elements.

1.1.3. General Form of a Matrix. An arrangement

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

is a matrix having 'm' rows and 'n' columns. The matrix may be written as $A = [a_{ij}]_{m \times n}$. The representation is called general form of a matrix.

Observe that the element a_{11} lies in the Ist row and Ist column. The element a_{23} lies in IInd row and 3rd column. The element a_{ij} lies in i-th row and j-th column. **The first suffix of every entry indicates the row and IInd suffix indicates the column in which the element lies.**

Remark: A matrix is marely an arrangement and has no numerical value.

1.1.4. Types of Matrices

I. Square Matrix. A matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix if the number of rows and columns in the matrix are equal *i.e.*, if $m = n$.

A square matrix having 'n' rows is called a matrix of order 'n' or 'n' square matrix *e.g.*,

the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$ is a square matrix of order 3.

II. Diagonal Elements of a Matrix : An element a_{ij} of a **square** matrix $A = [a_{ij}]_{n \times n}$ is said to be a diagonal element if $i = j$ *i.e.*, the elements $a_{11}, a_{22}, \ldots$ are diagonal elements.

III. Principal Diagonal : The places along which the diagonal elements lie is called the principal diagonal of a square matrix.

IV. Row Matrix : A matrix having only one row is called a row matrix *e.g.*, the matrix $A = [1\ 3\ 7]_{1 \times 3}$ has one row and three columns is a row matrix.

V. Column Matrix : If a matrix has only one column, then it is called a column matrix

e.g., $\begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}_{3 \times 1}$ is a column matrix.

VI. Zero Matrix or Null Matrix : A matrix $A = [a_{ij}]_{m \times n}$ is called a zero matrix or null matrix if $a_{ij} = 0\ \forall\ i$ and j *i.e.*, all the entries of the matrix A are zero *e.g.*,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is a zero matrix.}$$

VII. Diagonal Matrix : A **square matrix** $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if $a_{ij} = 0\ \forall\ i \neq j$ *i.e.*, all the non-diagonal entries are zero *e.g.*,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ are diagonal matrices.}$$

VIII. Scalar Matrix : A **diagonal matrix** in which all the diagonal entries are same is called a scalar matrix *e.g.*,

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \text{ is a scalar matrix.}$$

IX. Identity or Unit Matrix : A **diagonal matrix** in which all the diagonal entries are 1, is called an identity or a unit matrix. An identity matrix of order 'n' is denoted by I_n *e.g.*,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ etc.}$$

X. Upper Triangular Matrix : A **square matrix** $A = [a_{ij}]_n$ is called an upper triangular matrix if $a_{ij} = 0\ \forall\ i > j$ *i.e.*, all the elements below the principal diagonal are zero *e.g.*,

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \text{ is an upper triangular matrix.}$$

XI. Lower Triangular Matrix : A **square matrix** $A = [a_{ij}]_n$ is called a lower triangular matrix if $a_{ij} = 0 \ \forall \ i < j$ *i.e.,* all the elements above the principal diagonal are zero *e.g.,*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 5 & 5 & 2 \end{bmatrix} \text{ is a lower triangular matrix.}$$

XII. Comparable Matrices. Two matrices A and B are said to be **comparable** if they are of the same order *e.g.,*

The matrices $A = \begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix}_{2 \times 3}$ and $B = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 6 \end{bmatrix}_{2 \times 3}$ are comparable.

XIII. Equal Matrices. Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are **equal** if they are of the same order and their respective entries are equal *i.e.,* if $m = p$ and $n = q$ and $a_{ij} = b_{ij} \ \forall \ i$ and j.

e.g., the two matrices $A = \begin{bmatrix} a & b & 4 \\ c & x & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 4 \\ 6 & 7 & 6 \end{bmatrix}$ are equal if $a = 0$, $b = 3$, $c = 6$, $x = 7$.

Example 1. *What is the order of the matrix given below ?*

$$A = \begin{bmatrix} 7 & 1 & 9 & -11 \\ 2 & 3 & 8 & 15 \\ -1 & -7 & -12 & 6 \end{bmatrix}$$

Write the elements a_{12}, a_{21}, a_{24}, a_{31}, a_{34} for the matrix A.

Sol. The given matrix A has three rows and four columns.

∴ The order of A is **3 × 4.**

a_{12} = element lying in Ist row and IInd column = **1**

a_{21} = element lying in IInd row and Ist column = **2**

a_{24} = element lying in IInd row and IVth column = **15**

a_{31} = element lying in IIIrd row and Ist column = **– 1**

a_{34} = element lying in IIIrd row and IVth column = **6.**

Example 2. *Construct a matrix $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = \dfrac{(i + 2j)^2}{2}$.*

Sol.
$$A = [a_{ij}]_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ given } a_{ij} = \frac{(i + 2j)^2}{2}$$

so
$$a_{11} = \frac{(1 + 2.1)^2}{2} = \frac{9}{2}, \ a_{12} = \frac{(1 + 2.2)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2 + 2.1)^2}{3} = \frac{16}{2} = 8, \ a_{22} = \frac{(2 + 2.2)^2}{2} = \frac{36}{2} = 18$$

so $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{9/2} & \mathbf{25/2} \\ \mathbf{8} & \mathbf{18} \end{bmatrix}.$$

Example 3. *Find the values of x, y, z if* $\begin{bmatrix} 2x+y & x-y \\ x-z & x+y+z \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ 2 & 8 \end{bmatrix}$.

Sol. We have $$\begin{bmatrix} 2x+y & x-y \\ x-z & x+y+z \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ 2 & 8 \end{bmatrix}.$$

By equality of matrices :

$$2x + y = 10 \quad ...(1)$$
$$x - y = -1 \quad ...(2)$$
$$x - z = 2 \quad ...(3)$$
$$x + y + z = 8 \quad ...(4)$$

$(1) + (2) \Rightarrow 3x = 9 \Rightarrow x = 3$

$\therefore$ $(2) \Rightarrow 3 - y = -1 \quad \therefore \quad y = 3 + 1 = 4$

$(3) \Rightarrow 3 - z = 2 \quad \therefore \quad z = 3 - 2 = 1$

$\therefore$ $\mathbf{x = 3, y = 4, z = 1.}$

Example 4. *Find x and y if the two matrices* $A = \begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix}$ *and* $B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$ *are equal.*

Sol. Since $$\begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

so there respective entries must be equal.

i.e.,
$$2x + 1 = x + 3 \quad ...(i)$$
$$3y = y^2 + 2 \quad ...(ii)$$
$$y^2 - 5y = -6 \quad ...(iii)$$

From (*i*), $x = 2$

From (*ii*), $y^2 - 3y + 2 = 0 \quad ...(iv)$

and from (*iii*), $y^2 - 5y + 6 = 0 \quad ...(v)$

Subtract (*iv*) from (*v*), we get

$$-2y + 4 = 0$$

$\Rightarrow$ $2y = 4 \Rightarrow \mathbf{y = 2}$

so $\mathbf{x = 2}$ and $\mathbf{y = 2.}$

EXERCISE 1.1

1. What is the order of the matrix A given below :

$$A = \begin{bmatrix} -7 & 8 & 6 & 5 \\ 2 & 7 & 11 & 17 \\ 3 & 9 & -6 & 14 \end{bmatrix}.$$

Write the elements $a_{21}, a_{23}, a_{14}, a_{34}, a_{12}$.

2. Write the type and order of the following matrices :

(*i*) $\begin{bmatrix} 2 & 3 & 6 \\ 0 & 8 & 7 \\ 1 & 2 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 21 \\ 3 \\ 6 \\ 8 \end{bmatrix}$ (*iii*) $[3 \ \ 4 \ \ 7]$

(*iv*) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (*v*) $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ (*vi*) $\begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix}$

(*vii*) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (*viii*) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (*ix*) $\begin{bmatrix} 5 & 0 & 0 \\ 6 & 18 & 0 \\ 10 & 0 & 11 \end{bmatrix}$.

3. Construct a 2 × 2 matrix A = $[a_{ij}]$ whose element a_{ij} is given by :

   (*i*) $a_{ij} = \dfrac{(i+j)^2}{2}$ (*ii*) $a_{ij} = \dfrac{(i-j)^2}{2}$ (*iii*) $a_{ij} = \dfrac{(i-2j)^2}{2}$.

4. If $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, find the values of a and b.

5. Find the values of a, b, c, d from the following matrix equations :

   (*i*) $\begin{bmatrix} 2a+5 & b+7 \\ 3c-8 & 3d+9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$.

Answers

1. 3 × 4 ; 2, 11, 5, 14, 8
2. (*i*) Square matrix, 3 × 3 (*ii*) Column matrix, 4 × 1
   (*iii*) Row matrix, 1 × 3 (*iv*) Zero matrix, 2 × 4
   (*v*) Scalar matrix, 3 × 3 (*vi*) Diagonal matrix, 2 × 2
   (*vii*) Unit matrix, 3 × 3 (*viii*) Unit matrix, 2 × 2
   (*ix*) Lower triangular matrix, 3 × 3
3. (*i*) $\begin{bmatrix} 2 & 9/2 \\ 9/2 & 8 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ (*iii*) $\begin{bmatrix} 1/2 & 9/2 \\ 0 & 2 \end{bmatrix}$.
4. $a = 2, b = 4$ or $a = 4, b = 2$
5. (*i*) $a = -2, b = -5, c = 3, d = -2$ (*ii*) $a = 1, b = 2, c = 3, d = 4$.

1.2. OPERATIONS ON MATRICES

1.2.1. Scalar Multiplication. Multiplication of a matrix by a scalar.

Let A = $[a_{ij}]_{m \times n}$ be a matrix and let λ be a scalar we define

$$\lambda A = \lambda\, [a_{ij}]_{m \times n} = [\lambda a_{ij}]_{m \times n}$$

i.e., if we multiply a matrix by some constant 'λ' say, then every entry of the matrix is multiplied by λ *e.g.*,

Let $$A = \begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix}, \text{ then } 9A = 9\begin{bmatrix} 2 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9\times 2 & 9\times 5 \\ 9\times 2 & 9\times 3 \end{bmatrix} = \begin{bmatrix} 18 & 45 \\ 18 & 27 \end{bmatrix}.$$

Note that common is also taken from every entry of the matrix.

1.2.2. Negative of a Matrix. The negative of a matrix A = $[a_{ij}]_{m \times n}$ is $[-a_{ij}]_{m \times n}$ and is denoted by – A *i.e.*, –ve of a matrix is obtained by multiplying every entry by '–1' or changing the sign of every entry for example :

Let $$A = [a_{ij}]_{2 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

We define $$-A = (-1)\,A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$$

1.2.3. Addition of Matrices. We can add two matrices only if they are of the same order. Sum of two matrices $A = [a_{ij}]_{m \times n}$, and $B = [b_{ij}]_{m \times n}$ is obtained by adding the respective entries of the two matrices *i.e.*, $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$.

For example, if $A = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 0 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 6 & 8 \\ 7 & 0 & 8 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 8 \\ 7 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2+0 & 3+6 & 6+8 \\ 4+7 & 0+0 & 9+8 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 14 \\ 11 & 0 & 17 \end{bmatrix}.$$

If $C = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, then neither A + C nor B + C is defined.

1.2.4. Properties of Matrix Addition and Scalar Multiplication

(*i*) $\lambda(A + B) = \lambda A + \lambda B$ *i.e.*, scalar multiplication is distributive.

(*ii*) $(\lambda.l)A = \lambda(lA)$.

(*iii*) $1.A = A$.

(*iv*) If A and B are matrices of the same order, then A + B = B + A *i.e.*, matrix addition is commutative.

(*v*) If A, B, C are matrices of the same order, then (A + B) + C = A + (B + C) *i.e.*, matrix addition is associative.

(*vi*) If A is any matrix then A + O = A = O + A , where 'O' is a zero matrix of order same as that of A.

Example 1. *If* $A = \begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$, *find 7A + 5B.*

Sol. $$7A + 5B = 7\begin{bmatrix} 3 & 8 & 11 \\ 6 & -3 & 8 \end{bmatrix} + 5\begin{bmatrix} 1 & -6 & 15 \\ 3 & 8 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 7\times3 & 7\times8 & 7\times11 \\ 7\times6 & 7\times-3 & 7\times8 \end{bmatrix} + \begin{bmatrix} 5\times1 & 5\times-6 & 5\times15 \\ 5\times3 & 5\times8 & 5\times17 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 56 & 77 \\ 42 & -21 & 56 \end{bmatrix} + \begin{bmatrix} 5 & -30 & 75 \\ 15 & 40 & 85 \end{bmatrix}$$

$$= \begin{bmatrix} 21+5 & 56-30 & 77+75 \\ 42+15 & -21+40 & 56+85 \end{bmatrix} = \begin{bmatrix} 26 & 26 & 152 \\ 57 & 19 & 141 \end{bmatrix}.$$

Example 2. *If* $A = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 11 \\ 6 & 1 \end{bmatrix}$, *then show that :*

(*i*) *A + B = B + A* (*ii*) *(A + B) + C = A + (B + C)*

(*iii*) *A + O = O + A = A* (*iv*) *A + (– A) = (– A) + A = O.*

Sol. (*i*) $$A + B = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 4+3 & 7+8 \\ 6+7 & 5+2 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 13 & 7 \end{bmatrix}$$

$$B + A = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 3+4 & 8+7 \\ 7+6 & 2+5 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 13 & 7 \end{bmatrix}$$

$\therefore$ **A + B = B + A.**

(*ii*) We have $A + B = \begin{bmatrix} 7 & 15 \\ 13 & 7 \end{bmatrix}$.

$$\therefore \quad (A + B) + C = \begin{bmatrix} 7 & 15 \\ 13 & 7 \end{bmatrix} + \begin{bmatrix} 8 & 11 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 7+8 & 15+11 \\ 13+6 & 7+1 \end{bmatrix} = \begin{bmatrix} 15 & 26 \\ 19 & 8 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 11 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 3+8 & 8+11 \\ 7+6 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 19 \\ 13 & 3 \end{bmatrix}$$

$$\therefore \quad A + (B + C) = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} 11 & 19 \\ 13 & 3 \end{bmatrix} = \begin{bmatrix} 4+11 & 7+19 \\ 6+13 & 5+3 \end{bmatrix} = \begin{bmatrix} 15 & 26 \\ 19 & 8 \end{bmatrix}.$$

$\therefore$ **(A + B) + C = A + (B + C).**

(*iii*) Since A is of order 2 × 2, we take $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so that A + O and O + A may be defined.

$$A + O = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4+0 & 7+0 \\ 6+0 & 5+0 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} = A$$

$$O + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 0+4 & 0+7 \\ 0+6 & 0+5 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} = A$$

$\therefore$ **A + O = O + A = A.**

(*iv*) $A = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} \Rightarrow -A = \begin{bmatrix} -4 & -7 \\ -6 & -5 \end{bmatrix}$

$$\therefore A + (-A) = \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} + \begin{bmatrix} -4 & -7 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} 4+(-4) & 7+(-7) \\ 6+(-6) & 5+(-5) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$(-A) + A = \begin{bmatrix} -4 & -7 \\ -6 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} -4+4 & -7+7 \\ -6+6 & -5+5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore$ **A + (– A) = (– A) + A = O.**

Example 3. *If* $\begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix} + X = \begin{bmatrix} 2 & -7 \\ 1 & 8 \end{bmatrix}$, *find matrix X.*

Sol. Let $A = \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 & -7 \\ 1 & 8 \end{bmatrix}$.

so A + X = B.

$$\Rightarrow \quad X = B - A = \begin{bmatrix} 2 & -7 \\ 1 & 8 \end{bmatrix} - \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 2-1 & -7+5 \\ 1-6 & 8-7 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -5 & 1 \end{bmatrix}.$$

Example 4. *If* $A = \begin{bmatrix} 2 & -2 \\ 4 & -2 \\ -5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$, *find the matrix X, such that 2A + 3X = 5B.*

Sol. We have 2A + 3X = 5B. $\therefore$ $3X = 5B - 2A = 5\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix} - 2\begin{bmatrix} 2 & -2 \\ 4 & -2 \\ -5 & 1 \end{bmatrix}$

$$\Rightarrow \quad 3X = \begin{bmatrix} 40 & 0 \\ 20 & -10 \\ 15 & 30 \end{bmatrix} - \begin{bmatrix} 4 & -4 \\ 8 & -4 \\ -10 & 2 \end{bmatrix} = \begin{bmatrix} 40-4 & 0+4 \\ 20-8 & -10+4 \\ 15+10 & 30-2 \end{bmatrix} = \begin{bmatrix} 36 & 4 \\ 12 & -6 \\ 25 & 28 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{3}\begin{bmatrix} 36 & 4 \\ 12 & -6 \\ 25 & 28 \end{bmatrix} = \begin{bmatrix} 12 & 4/3 \\ 4 & -2 \\ 25/3 & 28/3 \end{bmatrix}.$$

Example 5. *Express* $4\begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$ *as a single matrix.*

Sol. $4\begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 12 \\ 4 & -16 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ 2 & -20 \end{bmatrix}.$$

Example 6. *Find X and Y, if* $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ *and* $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Sol. We have $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ ...(1)

and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ...(2)

Adding equations (1) and (2), we get

$$X + Y + X - Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow \quad 2X + O = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \quad \Rightarrow \quad X = \tfrac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 10/2 & 0/2 \\ 2/2 & 8/2 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}$$

$$\therefore \quad (1) \Rightarrow \quad \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \quad \Rightarrow \quad Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}.$$

Remark. The matrix Y can also be found by considering (1) – (2).

Example 7. *Find X and Y if* $2X + Y = \begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix}$ *and* $X - 2Y = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

Sol. We have $2X + Y = \begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix}$ ...(1)

and $X - 2Y = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ ...(2)

(1) × 2 $\Rightarrow$ $4X + 2Y = 2\begin{bmatrix} 4 & 4 & 7 \\ 7 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 14 \\ 14 & 6 & 8 \end{bmatrix}$ ...(3)