Eastern Economy Edition

COMPLEX VARIABLES

Theory and Applications

H. S. KASANA

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SECOND EDITION

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COMPLEX VARIABLES: Theory and Applications, Second Edition
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Preface

This book is an outcome of the lectures delivered by me at Birla Institute of Technology and Science (BITS), Pilani. However, writing of the text started in 1991 when I was on a visiting assignment at Uppsala University, Sweden. Since then, the process of teaching the course has been continued at Thapar Institute of Engineering and Technology, Patiala. While teaching this course I felt the need of a suitable text on complex variables for imparting comprehensive knowledge of various concepts and problem-solving techniques to students.

This book is designed to meet the requirement of graduate and undergraduate students in the science and engineering disciplines. I have tried teaching the contents of this book as a core course in one semester with four hours per week. In order to write a relatively brief text on a vast subject, I had to make a number of choices about inclusion and exclusion of certain topics. In spite of this constraint, a wide selection of topics has been achieved. The basic concepts and fundamental techniques have been emphasized while highly specialized topics and methods have not been given prominence.

The level of the text assumes that the reader is acquainted with elementary real analysis. Enough care has been taken so that the reader may attempt different type of problems. Each chapter consists of a number of unsolved problem sets. Standard problems have been specially designed for this text. Geometric interpretation of the results wherever necessary has been inducted for making the analysis more accessible. The book has been written in a fairly sequential manner so that understanding of most chapters requires familiarity with the preceding ones. However, Chapter 9 may be taken up directly after Chapter 4 without any difficulty.

This book has been made possible through academic interactions with Prof. M.G. Nadkarni (University of Mumbai), Prof. G.S. Srivastava (IIT-Roorkee), Prof. Matts Essén (Uppsala University, Sweden). During the course of writing the text I have received valuable contributions from my colleagues, Profs. B. Singh, P. Singh, M.C. Datta and S.P. Yadava, all of BITS, Pilani.

At this moment I cannot forget my old friends, G. Indira, Rashmi Mishra, Kompella Kalyani and Harendra Singh Nehra, who all requested me a number of times to contribute a book on complex variables. To honour the sentiments of these friends this book has been so fashioned that it caters to those who use complex variables and need its applications at some stage or the other in their professional pursuits.

More solved and unsolved problems have been added to the second edition of the book. I have modified and improved the proofs of various results in view of the latest developments in the field of complex analysis.

H.S. Kasana

List of Symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{I}$	set of integers
$\mathbb{Q}$	set of rational integers
$\mathbb{R}$	real number system
$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$	set of all ordered pair of real numbers
$\mathbb{C}$	complex numbers system
$\text{Re } z$	real part of $z = x + iy$
$\text{Im } z$	imaginary part of $z = x + iy$
$\|z\|$	modulus of $z = x + iy$
$\arg z$	set of all real values of $\theta : z = re^{i\theta}$
Arg	values of θ: $-\pi < \theta \le \pi$
$\mathbb{I}_0$	set of nonzero integers
$\mathbb{I}_+$	set of positive integers
$\mathbb{R}_+$	set of positive real numbers
$N_\delta(z_0) = \{z : \|z - z_0\| < \delta\}$	δ-neighbourhood of z_0
$N_\delta(z_0)\backslash z_0 = \{z : 0 < \|z - z_0\| < \delta\}$	deleted δ-neighbourhood of z_0
∂S	boundary of the set S
$\bar{S}$	clouser of the set S, $\bar{S} = S \cup \partial S$
D/Ω	domain
$\sup S$	least upper bound of the set S
$\inf S$	greatest lower bound of the set S
$u \equiv u(x, y)$	real component of $f = u + iv$
$v \equiv (x, y)$	imaginary component of $f = u + iv$
$\gamma \equiv \gamma(t) = x(t) + iy(t)$	path in the complex plane
$\Rightarrow$	implies (gives)
$\Leftarrow$	implied by
$\Leftrightarrow$	if and only if
$\frac{\partial}{\partial z}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$
$\frac{\partial}{\partial \bar{z}}$	$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$
f_z	$\frac{\partial f}{\partial z}$
$f_{\bar{z}}$	$\frac{\partial f}{\partial \bar{z}}$
$\ln x$	natural logarithm
lim sup	limit superior
lim inf	limit inferior
$\exists$	there exists
$[z_1 z_2]$	line segment joining z_1 and z_2

$\gamma_1 + \gamma_2$	sum of the paths γ_1 and γ_2
e^z	exponential function of z
$\log z = \ln\lvert z\rvert + i \arg z$	logarithmic function of z
$\operatorname{Log} z = \ln z + i \operatorname{Arg} z$	principal logarithmic function of z
$\eta(\gamma, a)$	winding number of a path γ with respect to a
$\operatorname{Res}_{z=a} f(z)$	residue of $f(z)$ at $z = a$
PV	Principal Value (Cauchy)
O	big oh
o	small oh
$F(p)$	Laplace transform of f
$n(r)$	number of zeros of f in $\lvert z\rvert \le r$
ρ	order of an entire function
σ	type of an entire function

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Algebra of Complex Numbers

This chapter starts with the theme how the real number system is extended to the complex number system. Various inequalities, polar and exponential forms and powers and roots of the complex numbers are included. A brief exposition of point set topology of the complex plane follows. At the end of the chapter, the extended complex plane is introduced.

1.1 COMPLEX NUMBERS

The equation $x^2 + 5 = 0$ has no solution in the real number system, i.e. there does not exist any real number which satisfies $x^2 = -5$. This prompted mathematicians to enlarge the real number system. However, it seems quite reasonable to impose the requirement that any number system suitable for computational purpose should permit us to solve this equation.

For solving such equations the real number system was enlarged to complex number system and G. Cardano was the first Italian mathematician who found the formula for cubic equations. The term *complex number* was introduced by C.F. Gauss, a German mathematician. Later on, the subject was enriched by the original work of A.L. Cauchy, B. Riemann, K. Weierstrass, and others.

Here, we develop the concept of complex numbers through algebra. The counting numbers $\{1, 2, 3, \ldots\}$, known as natural numbers, denoted by $\mathbb{N}$ constituted the first set conceived by civilization. It was observed that the difference of two natural numbers may not be a natural number. Hence the system $\mathbb{N}$ was extended to the set of integers $\mathbb{I}$ by taking $\mathbb{N} \times \mathbb{N}$ and defining the operations of addition and subtraction. Further, from the viewpoint of including the operation of division, the integers were enlarged by considering $\mathbb{I} \times \mathbb{I}$ to $\mathbb{Q}$, the set of rational numbers. In a similar fashion, the real number system $\mathbb{R}$ is extended to the complex number system by defining appropriate operations on $\mathbb{R} \times \mathbb{R} \equiv \mathbb{R}^2$.

Consider $\mathbb{R}^2$, the set of all ordered pairs of real numbers (x, y) defined by

$$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}.$$

Here 'ordered' means that (x, y) and (y, x) are different unless $x = y$ and, (x, y), a nonzero element in $\mathbb{R}^2$, means that at least one of the members of the ordered pair is nonzero.

We define the operations of addition $(+)$ and multiplication $(\cdot)$ in $\mathbb{R}^2$ as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{1.1}$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2). \tag{1.2}$$

The set $\mathbb{R}^2$ forms a commutative group with respect to addition $(+)$ defined in (1.1). Also, the set of all nonzero elements in $\mathbb{R}^2$ forms a commutative group with respect to multiplication $(\cdot)$ defined in (1.2). The reader can prove these statements easily.

Proposition 1. $\mathbb{R}^2$ forms a field under addition and multiplication as defined in (1.1) and (1.2), respectively.

Proof. (a) $(\mathbb{R}^2, +)$ is a commutative group.

(b) $(\mathbb{R}^2 \backslash \{(0, 0)\}, \cdot)$ is a commutative group.

It remains to show that $(\cdot)$ is distributive over $(+)$. We observe

$$\begin{aligned}(x_1, y_1) \cdot [(x_2, y_2) + (x_3, y_3)] &= (x_1, y_1) \cdot (x_2 + x_3, y_2 + y_3) \\ &= (x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3, x_1y_2 + x_1y_3 + y_1x_2 + y_1x_3) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) + (x_1x_3 - y_1y_3, x_1y_3 + y_1x_3) \\ &= (x_1, y_1) \cdot (x_2, y_2) + (x_1, y_1) \cdot (x_3, y_3).\end{aligned}$$

Similarly, it can be argued that

$$[(x_1, y_1) + (x_2, y_2)] \cdot (x_3, y_3) = (x_1, y_1) \cdot (x_3, y_3) + (x_2, y_2) \cdot (x_3, y_3).$$

Definition 1. A complex number is defined as the ordered pair (x, y) of real numbers, $z = (x, y)$ satisfying the following rules for addition and multiplication:

$$\begin{aligned}z_1 + z_2 &= (x_1 + x_2, y_1 + y_2), \\ z_1z_2 &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2),\end{aligned}$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Also, (i) two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal $\Leftrightarrow x_1 = x_2$ and $y_1 = y_2$; (ii) a complex number $z = (x, y)$ is nonzero if at least one of the numbers x and y is nonzero.

In view of definition of complex numbers and Proposition 1, it is obvious that the set of all complex numbers $\mathbb{C}$ forms a field with respect to addition and multiplication. Actually, this is a translation of Proposition 1. Although, it is not our purpose to study fields or any other algebraic structure in detail, it is worthwhile to prove that some familiar properties of $\mathbb{C}$ are the consequences of field axioms.

It is trivial to prove that for any real numbers a and b, we have

$$(a, 0) + (b, 0) = (a + b, 0); \qquad (a, 0) \cdot (b, 0) = (ab, 0).$$

This motivates the idea that complex numbers of the form $(a, 0)$ have the same arithmetic properties as the corresponding real numbers a. We can therefore identify the ordered pair $(a, 0)$ by the real number a. This identification gives us the real field as a subfield of the complex field.

Definition 2. The imaginary unit i (*iota*) is defined as

$$i = (0, 1).$$

Proposition 2. $i^2 = -1$.

Proof. $i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (0 - 1, 0 + 0) = (-1, 0) \equiv -1.$

Proposition 3. Every complex number $z = (x, y)$ can be written as $z = x + iy$ or $x + yi$, where x and y are real and imaginary parts of z and denoted by Re z and Im z, respectively.

Proof. We have

$$z = (x, y) = (x, 0) + (0, y) = (x, 0) + (0, 1)\,(y, 0) = x + iy.$$

Similarly, $z = x + yi$.

With the introduction of i, Definition 1 for addition and multiplication of complex numbers can be translated as

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$
$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

From now onwards $x + iy$ or $x + yi$ will have the same meaning. For $z = (x, y)$, the additive inverse will be defined by $-z = (-x, -y)$ since $z + (-z) = (-z) + z = 0$. Further, if $z \neq 0$, then the multiplicative inverse is given by solving the equations

$$(x + iy)(a + ib) = 1 = (a + ib)(x + iy)$$

or

$$ax - by = 1 \quad \text{and} \quad bx + ay = 0.$$

These linear simultaneous equations have the unique solution

$$a = \frac{x}{x^2 + y^2} \qquad \text{and} \qquad b = -\frac{y}{x^2 + y^2}.$$

Denoting $a + ib$ by z^{-1}, we get

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right) \quad \text{or} \quad z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}. \tag{1.3}$$

Consequences of $\mathbb{C}$ to be a field are:

(i) Additive identity is unique $z = 0 \equiv (0, 0)$.

(ii) Additive inverse of $z = (x, y)$ as $z = (-x, -y)$ is unique.

(iii) Cancellation laws hold for addition;

$$z + z_1 = z + z_2 \Rightarrow z_1 = z_2,$$
$$z_1 + z = z_2 + z \Rightarrow z_1 = z_2.$$

(iv) The multiplicative identity $z = 1 \equiv (1, 0)$ is unique.

(v) For $z \neq 0$, the multiplicative inverse z^{-1} is unique.

(vi) If $z \neq 0$, then

$$zz_1 = zz_2 \Rightarrow z_1 = z_2,$$

$$z_1 z = z_2 z \Rightarrow z_1 = z_2.$$

(vii) If $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$.

Subtraction

This is the inverse process of addition, z_2 is subtracted from z_1 as

$$z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2).$$

Division

If $z_2 \neq 0$, then division by z_2 is defined as

$$\frac{z_1}{z_2} = z_1 z_2^{-1}.$$

In this way, for $z_1 = 1$, $1/z_2 = z_2^{-1}$, we can write

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = z_1 \frac{1}{z_2}.$$

So, we have the following simple results:

$$(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}, \qquad \text{since } (z_1 z_2)(z_1^{-1} z_2^{-1}) = 1 \text{ and } (z_1^{-1} z_2^{-1})(z_1 z_2) = 1,$$

$$\frac{1}{z_1 z_2} = (z_1 z_2)^{-1} = z_1^{-1} z_2^{-1},$$

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2) z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3}.$$

Now, $\mathbb{R}^2$ has been identified as $\mathbb{C}$ provided addition and multiplication of elements of $\mathbb{R}^2$ are defined by (1.1) and (1.2). For definition, other than (1.1) and (1.2), we shall not permit $\mathbb{R}^2$ to be identified by $\mathbb{C}$. Note that

$$\text{Re } z_1 + \text{Re } z_2 = \text{Re}(z_1 + z_2), \tag{1.4}$$

$$\text{Im } z_1 + \text{Im } z_2 = \text{Im}(z_1 + z_2). \tag{1.5}$$

Geometric representation

Every complex number can be represented geometrically as a point in xy-plane. Now, we shall call this plane as complex plane or argand diagram. Introducing a rectangular coordinate system in the plane, we can identify the complex number $z = x + iy$ with the point $P = (x, y)$. Clearly, the set of all real numbers $(x, 0)$ corresponds to the x-axis, called the real axis, and the set of all purely imaginary numbers $(0, y)$ corresponds to the y-axis, and hence called the imaginary axis, while the set of all imaginary numbers corresponds

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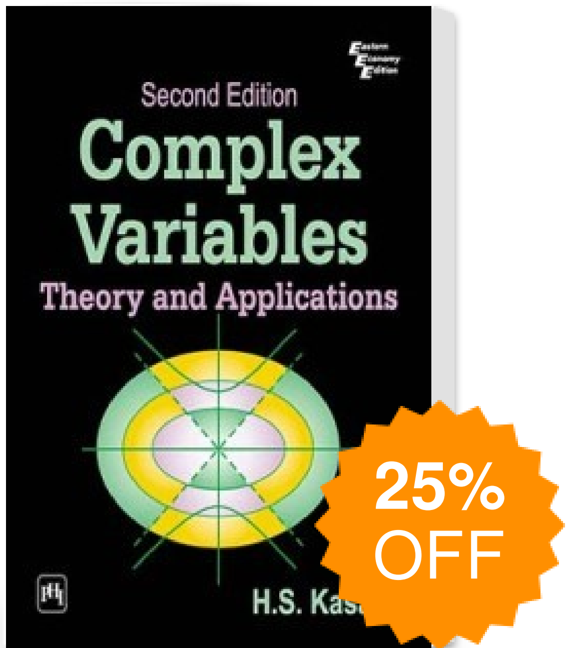

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