Introduction to
PARTIAL DIFFERENTIAL EQUATIONS
THIRD EDITION

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This book is dedicated with affection and gratitude to the memory of my respected Father
(Late) KOMMURI VENKATESWARLU
and
to my respected Mother
SHRIMATI VENKATARATNAMMA
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Preface

The objective of this third edition is the same as in previous two editions: to provide a broad coverage of various mathematical techniques that are widely used for solving and to get analytical solutions to Partial Differential Equations of first and second order, which occur in science and engineering. In fact, while writing this book, I have been guided by a simple teaching philosophy: An ideal textbook should teach the students to solve problems. This book contains hundreds of carefully chosen worked-out examples, which introduce and clarify every new concept. The core material presented in the second edition remains unchanged.

I have updated the previous edition by adding new material as suggested by my active colleagues, friends and students.

Chapter 1 has been updated by adding new sections on both homogeneous and non-homogeneous linear PDEs, with constant coefficients, while Chapter 2 has been repeated as such with the only addition that a solution to Helmholtz equation using variables separable method is discussed in detail.

In Chapter 3, few models of non-linear PDEs have been introduced. In particular, the exact solution of the IVP for non-linear Burger’s equation is obtained using Cole–Hopf function.

Chapter 4 has been updated with additional comments and explanations, for better understanding of normal modes of vibrations of a stretched string.

Chapters 5–7 remain unchanged.

I wish to express my gratitude to various authors, whose works are referred to while writing this book, as listed in the Bibliography. Finally, I would like to thank all my old colleagues, friends and students, whose feedback has helped me to improve over previous two editions.

It is also a pleasure to thank the publisher, PHI Learning, for their careful processing of the manuscript both at the editorial and production stages.

Any suggestions, remarks and constructive comments for the improvement of text are always welcome.

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