MATHEMATICAL PHYSICS

Fourth Edition

B D Gupta

VIKAS

MATHEMATICAL PHYSICS

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FOURTH EDITION

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First Edition, 1978
Fourth Edition, 2010

Printed at Millenium Offset Pvt. Ltd., Delhi-110028

PREFACE TO THE FOURTH EDITION

I have once again availed the opportunity of revising this work, designed to suit the requirements of students of Applied Mathematics, Physics and Technical courses at both undergraduate and postgraduate levels. In the Third Edition, as required by the UGC syllabus as well as syllabi of several other universities of India many topics such as Conformal Mapping in Complex Analysis, Inversion of Complex Matrices, Error Functions, Factorial Functions, etc., and specially the solution of Schröedinger's Wave-equation in Quantum Mechanics, had been added. However a need was still felt to add some more useful topics to enhance the utility of the work, without going into competitive spirit to make it bulky as done at the primary levels, where there is almost a kind of competition to make the schoolbag heavier by putting into it, even bricks or rough and tough materials taken from any corner, without thinking whether the fabricated material matches the needs of the reader who may be an ordinary student or a researcher i.e., if the author himself is competant enough to undergo the research work.

In the present revised work, I have mainly touched upon the following topics for new additions and alterations:

(*i*) *In Vectors*: Transformation of Coordinates of a vector on a change of bases, Rankine's Theorem, Helmholtz Theorem (*i.e.*, rigorous proof) and a few problems.

(*ii*) *In Matrices*: A criterion on Solutions of Simultaneous Equations, some more methods of Matrix Inversion, other elegant methods of matrix-inversion along with the general eigenvalue problem and a few problems.

(*iii*) *In Tensors*: Einstein Gravitational Equation and Equation of Planetary Orbits.

(*iv*) *In Complex Analysis*: Natural Boundary Theorem (Lambert's Series), a new article on Convergence of Integrals, Series and Products along with uniform and Absolute Convergence, and a few problems.

(*v*) *In Beta, Gamma and Error Functions*: Euler's product for $\Gamma(n)$ with different forms, convergence of Gamma Integrals, Values of $\Gamma'\left(\frac{1}{2}\right)$ and $\Gamma'(1)$, convergence of Beta Integrals, Orthogonal Sets of Functions with Mean Square Error and Gram-Schmidt Ortho-Normalization and a few problems.

(*vi*) *In Differential Equations*: Euler's Method with modifications, Taylor's Series Method and Runge's Method for solving First Order Differential Equations.

(*vii*) *In Harmonics*: Some alternative approaches, Weber Bessel Functions, Neuman-Bessel Functions, Weber-Schlafli Functions, Lommet Theorem, Kelvin's Functions, Various Transformations, alternative solutions to Hermite Equation, other form of Hermite Polynomial along with Integral representation, Explanatory note on Integral Property of Laguerre Polynomials, altier to

Orthogonality of Laguerre Polynomials, Heaviside Unit Function, Hurwitz Formula, Debye Function and a few problems.

(*viii*) *In Appendices*: Approximate solutions of Algebraic and Transcendental Equations containing Graphic Method, Analytic Method, Newton-Raphson Method, Iterative Method (Linear Iteration method) and Regula-Falsi Method with illustrations.

References to examination papers have been at relevant places.

For this revision, I have freely consulted the contributions of Indian and foreign authors, in the form of the titles as *Mathematical Physics*, *Quantum Mechanics*, *Complex Analysis*, *Numerical Analysis* and *Mathematical Methods* with special functions. I am deeply grateful to the authors and publishers of all these books and my heartiest thanks to all of them.

I am also very thankful to the editorial team at Vikas Publishing House for taking keen interest in publishing the revised work well in time and in an attractive format.

I do hope that the work in its present form will be more useful to the readers for whom it is intended.

Although I have tried my best to make the work free from errors and omissions, but if the readers still find some misprints and take the pain of making me aware of them, then I shall feel personally obliged to them.

Any suggestions for further improvement of the work, such as the addition of any new topic or topics, will be thankfully accepted and executed.

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PREFACE TO THE FIRST EDITION

It was in the year 1965, that myself and my colleague BS Rajput, wrote a book entitled *Mathematical Physics*, to meet the requirements of Honours, Postgraduate and Engineering students of various Indian universities. Since then the same work has appeared in six editions with the previous publisher. But for many reasons, we could not continue the co-authorship.

Here I would like to offer my sincere thanks to my former co-author and previous publisher for giving their consents to get it revised and published independently.

Now, keeping in mind the needs of the changing syllabi of various universities and the requirements of changing knowledge which almost doubles in every decade, I prepared this work myself about five years back, in view of new and advanced standpoint. But owing to some unavoidable circumstances this could not be published earlier, and so some of its portions have been further revised in order to render it up-to-date, hence more useful.

The value and scope of the present work have obviously been considerably increased, because, in the first place, several portions contributed by myself, in all my books written with different co-authors, have been reproduced in a more expository style and in the second place many more portions have been rewritten. The entire matter has been rearranged and many new topics have been added. Every section has been supplemented with a large number of worked-out problems and a set of additional miscellaneous problems selected from the question papers set in various university examinations.

I do not claim any originality of matter and this is at best a compiled work, with a novel presentation. The subject matter has been so arranged that even a layman can understand how to apply the mathematical operations to the problems of Physics. During the preparation of this work, I have freely consulted a number of books on Mathematics and Physics written by foreign and Indian authors. It goes without saying that I am deeply indebted to all of them, although, I am sorry not to acknowledge my gratitude to them individually—the number being too big to be accommodated in the little space that can be spared for that purpose in a work of the nature.

While revising the Volume, I have developed chapter one on 'Vectors' from all my contributions to our 'Vector Analysis', 'Vector-Calculus' and 'Elements of Mechanics', chapter two on 'Matrices' from my contributions to our 'Mathematical Physics', chapter three on 'Tensors' from my contributions to 'Relativistic Mechanics' after the publication of which a number of sections on tensors were taken to our 'Mathematical Physics', chapter four on 'Group Theory' (written a new); chapters five to ten on 'Complex Variables', 'Beta', Gamma and Error Functions', 'Differential Equations', 'Harmonics', 'Fourier Series', 'Integrals and Transforms' and 'Laplace's Transforms' from my contributions to our 'Mathematical Physics'. Chapters eleven to fourteen on 'Hankel Transforms', 'Diffusion, Wave and Laplace's Equation', 'Maxwell's Electromagnetic

Field Equations' and 'Special Theory of Relativity' have been written quite new. Chapter fifteen on 'Statistical Probability' has been developed from my contributions to our 'Mathematical Statistics'. In the end, the three Appendices A, B, C on 'Some formulated results in Basic Mathematics', 'Asymptotic Expansion of Error Function', and 'Character Tables in Group Theory' have been added in order to enhance the utility of the work.

In the preparation of Appendix C, I have been much guided by 'Elements of Group Theory for Physicists' by A. W. Joshi, Published by Wiley Eastern Ltd., New Delhi, I acknowledge my indebtedness to all my past and present co-authors, chiefly my colleagues Messrs O.P. Gupta, H.C. Sharma, J.P. Agarwal, Satya Parkash (formerly a student of mine), and the publisher M/s Kedar Nath Ram Nath, Meerut.

My thanks are also due to A.W. Joshi of the Institute of Advanced Studies, Meerut University and to Naresh Kumar, my colleague and a former pupil of mine, for their concrete suggestions during the preparation of this book. I would be failing in my duty if I do not acknowledge my deep gratitude to my colleagues P.C. Jain, M.P. Tyagi, B. Singh and V.P. Arora for not only rendering me their best help and cooperation but also encouraging and inspiring me all along.

Any suggestions for further improvement of the book from any corner will be thankfully accepted and executed.

B D Gupta

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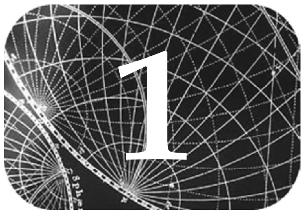

1 Vectors

1.1 INTRODUCTION

It is generally observed that there exist two types of physical measurements in applied mathematics, physics and mechanics: one involving *only magnitude and no direction* in the space of three dimensions, such as volume, mass, length, speed, temperature, potential, electric charge, etc., and the other involving *a definite direction in space associated with their magnitudes* such as velocity, acceleration, momentum, force, electric or magnetic field intensities, etc., the former being called *scalar quantities* or simply *scalars* and the latter, *vector quantities* or simply *Vectors*.

A little consideration will exhibit that the complete characterisation of a scalar quantity requires length and support, *i.e.*, a specified unit and a number stating how many times that unit is contained in that quantity, while the complete characterisation of a vector quantity requires length, support and sense, *i.e.*, a specified unit, a number stating how many times that unit is contained in that quantity and the statement of the direction.

Stating in a precise manner a *vector* means '*a directed line segment*'. In order words we can state that a vector is a quantity having direction as well as magnitude. In Astronomy a vector means an imaginary straight line that joins a planet moving round a centre (generally the focus of an elliptic orbit) to that centre.

1.2 REPRESENTATION OF VECTORS

Since a vector is the result of abstraction, its magnitude and direction may be represented by a line OP directed from the *initial point O* to the *terminal point P* and denoted by $\vec{OP}$. Here the length of vector $\vec{OP}$ denoted by $|\vec{OP}| = OP$ is called *magnitude* or *module* or *modulus* of the vector and the direction in space is indicated by an arrow head on the line.

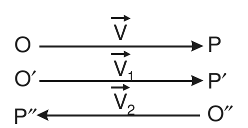

Fig. 1.1

In Fig. 1.1, the vector $\vec{OP}$ has been shown by $\vec{V}$ (or in Clarendon print by **V**) while its scalar magnitude is stated by V. Thus OP is the length of the vector **V**, while the line of indefinite length of which the directed line segment $\vec{OP}$ is only a part is the support of **V** and the sense is from O to P.

It should be noted that formulation of a law of physics in terms of vectors is however independent of the choice of axes of reference, *i.e.*, the vector representation has a physical content without reference to any coordinate system.

1.3 KINDS OF VECTORS

Equal vectors. Two given vectors may be equal only when they have the same magnitude and the same direction, *i.e.*, the given two vectors are equal if and only if they have the same or parallel support with equal length and the same sense. For example in Fig. 1.1, we have

$$\mathbf{V}\ (=\overrightarrow{OP}) = \mathbf{V}_1\ (=\overrightarrow{O'P'}) = -\mathbf{V}_2\ (=\overrightarrow{O''P''})$$

where $\mathbf{V}_1$ and $\mathbf{V}_2$ have the same scalar magnitude as $\mathbf{V}$ but $\mathbf{V}_1$ has the same and $\mathbf{V}_2$ the opposite sense to that of $\mathbf{V}$.

Null vector. A vector having the initial and the terminal points coincident is termed as a *zero vector* or a *null vector*. Thus a null vector has its module zero.

Unit vector. A vector having its modulus as unity is called a *unit vector.*

If $\mathbf{a}$ is a vector and 'a' its modulus, then unit vector $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$ (read as 'a hat' or 'a caret') defined as

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{a}$$

Polar vectors. The line vectors representing the quantities like force, velocity, etc., in which merely a linear action in a particular direction is involved, are termed as *polar* vectors.

Axial vectors. The line vectors representing the quantities like angular velocity, angular acceleration, etc., in which some rotational action is involved about an axis and which are drawn parallel to the axis of rotation in order that the magnitude of the quantity is determined by the length of the vector and the direction by the rule of right handed screw (*i.e.*, rotation being considered in clockwise direction), are termed as *axial* vectors.

Free vector. Evidently a vector can be represented by an infinite number of equal vectors by drawing parallel supports. Such a vector which can be transported from place to place such that it remains of the same magnitude and keep up the same direction is termed as a *free* vector. In fact a free vector is assumed to remain the same through transportation, irrespective of its position in space.

Localised or Line vector. We have defined that the value of a free vector depends only on its length and direction, but if it depends also on its position in space, *i.e.*, if a vector is restricted to pass through a given origin, then it is termed as a *localised* vector.

Collinear vectors. The vectors parallel to the same line, regardless of their magnitudes and sense of directions are termed as *collinear* vectors. In other words the vectors having the same or parallel supports are known as collinear vectors. Such vectors are parallel to each other and they may coincide in a special case. As such there exists a scalar ratio say λ between any two collinear vectors $\mathbf{a}$ and $\mathbf{b}$ of the form

$$\mathbf{b} = \lambda\,\mathbf{a}$$

which follows that one of the two collinear vectors can be expressed as the scalar multiple of the other.

Non-collinear vectors. The vectors whose directions are neither parallel nor coincident are said to be *non-collinear.*

Like vectors or co-directional vectors. The vectors which are collinear and have the same sense of directions, *i.e.*, the vectors directed in the same sense irrespective of their magnitudes are termed as *like* vectors.

Unlike vectors. The vectors which are collinear but have opposite sense of directions from each other are termed as *unlike* vectors.

Coplanar vectors. A system of vectors lying in the parallel planes or which can be made to lie in the same plane are said to be *coplanar* vectors. Evidently any two vectors are always coplanar.

Non-coplanar vectors. A system of vectors consisting of three or more vectors which cannot be made to lie in the same plane are called *non-coplanar* vectors.

Reciprocal vector. Any vector having its direction the same as that of a given vector **a**, but its magnitude as the reciprocal of the magnitude of **a** is termed as the *reciprocal* vector of **a** and written as $\mathbf{a}^{-1}$ or $\frac{1}{\mathbf{a}}$. As such

$$\mathbf{a}^{-1} = \frac{1}{a}\hat{\mathbf{a}} = \frac{\mathbf{a}}{a^2}\hat{\mathbf{a}} = \frac{\mathbf{a}}{a^2} \text{ (by definition of a unit vector).}$$

In this connection it is notable that the magnitude and so the reciprocal of the magnitude of a unit vector being unity, the unit vector is reciprocal to itself and it is said to be *self-reciprocal.*

Negative vector. The vector having the same magnitude as the vector **a** but opposite direction, is known as the *negative* of **a** and written as –**a**.

Position vector. If a vector $\vec{OP}$ specifies the position of a point relative to an arbitrarily chosen point O, then $\vec{OP}$ is called the *Position* vector of P with respect to O, the origin of vectors.

Problem 1. *If* {**a**, **b**, **c**} *is a right handed set, which of the following sets are right handed?*

(*i*) **a**, **c**, **b**; (*ii*) **b**, **c**, **a**; (*iii*) **b**, **a**, **c**; (*iv*) **c**, **a**, **b**; (*v*) **c**, **b**, **a.**

It is clear that the sets (*ii*) and (*iv*), *i.e.*, **b**, **c**, **a**; and **c**, **a**, **b** are right handed.

Problem 2. *Discuss the geometrical significance of* $a\mathbf{A} + b\mathbf{B} = 0$.

We have $a\mathbf{A} + b\mathbf{B} = 0$, a, b being scalars.

This can be written as $\mathbf{A} = -\frac{b}{a}\mathbf{B}$

$$= \lambda\mathbf{B} \quad \text{if} \quad \lambda = -\frac{b}{a}$$

i.e., **A** is expressible as a scalar multiple of **B** so that the vectors **A** and **B** are parallel or collinear.

1.4 ADDITION OF VECTORS

The charactertisation of process of summation is inherited in the composition of two or more displacements of a point. Suppose that we have two vectors **a** and **b** acting at a point O as shown in Fig. 1.2. Let $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$.

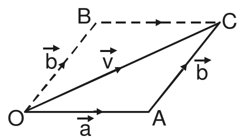

Fig. 1.2

Clearly the resultant effect of the vectors **a** and **b** is the same as that of their vector sum **v** obtained by setting off the vector **b** at the end of **a** and then joining the beginning of **a** to the end of **b**. This geometrical construction utilised to find the vector sum of two vectors **a** and **b** is known as *the parallelogram law of addition of vectors.*

Thus $\mathbf{v} = \vec{OC} = \vec{OA} + \vec{AC} = \mathbf{a} + \mathbf{b}$ ... (1)

A similar result follows by starting with **b** and setting off the vectors **a** on **b**, *i.e.*,

$$\mathbf{v} = \vec{OC} = \vec{OB} + \vec{BC} = \mathbf{b} + \mathbf{a} \quad \text{... (2)}$$

Conclusively the result of adding two co-initial vectors is the vector represented by the diagonal of the parallelogram having the two given vectors as its adjacent sides.

From (1) and (2), it follows that

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

i.e., the two vectors obey the **commutative law of addition**, according to which the vector sum of two vectors is independent of their order.

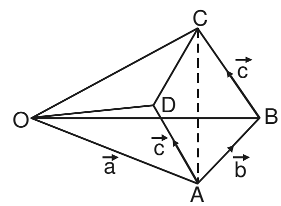

Fig. 1.3

We now propose to find the sum of three vectors say **a**, **b**, **c**. Let $\vec{OA} = \mathbf{a}$, $\vec{AB} = \mathbf{b}$, $\vec{BC} = \mathbf{c}$ as shown in Fig. 1.3. Then

$$\mathbf{v} = \vec{OC} = \vec{OA} + \vec{AB} + \vec{BC} = \mathbf{a} + \mathbf{b} + \mathbf{c}. \quad \text{... (3)}$$

Also

$$\mathbf{v} = \vec{OC} = \vec{OB} + \vec{BC}$$
$$= (\vec{OA} + \vec{AB}) + (\vec{BC})$$
$$= (\mathbf{a} + \mathbf{b}) + \mathbf{c}. \quad \text{... (4)}$$

Similarly $\mathbf{v} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ ... (5)

and $\mathbf{v} = (\mathbf{a} + \mathbf{c}) + \mathbf{b}$ ... (6)

It follows from (3), (4), (5) and (6) that

$$\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{c}) + \mathbf{b}.$$

i.e., the three vectors obey the **associative law of addition**, *according to which the vector sum of three vectors is independent of the mode in which component vectors are associated in different groups.*

In general, if there are *n* vectors **a**, **b**, **c**...**n**, then their vector sum **v** is given by

$$\mathbf{v} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \ldots + \mathbf{n}$$

1.5 SUBTRACTION OF VECTORS

If there are two vectors **a** and **b**, then

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}),$$

i.e., the subtraction of **b** from **a** may be regarded as the addition of **–b** to **a**. Thus, to substract **b** from **a**, reverse the direction of **b** and add to **a** (Fig. 1.4).

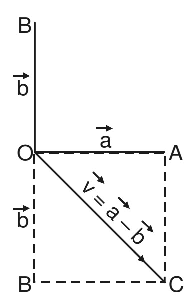

Fig. 1.4