Notes

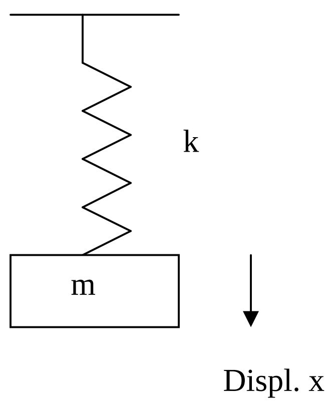
Mechanical Vibrations-II

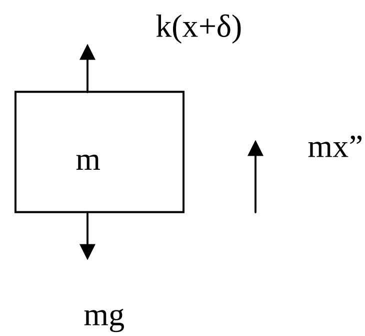
Mechanical Vibrations (ME 65)

Session 4 date: (6/3/07)

Undamped free vibrations

Single degree of freedom System





This consists of a single spring attached with a single mass. The Various ways in which the equation of motion is obtained are :

a) Newton’s Method b) Energy Method and c) Rayleigh Method

Newton’s Method

When a mass m is attached to a spring it deflects by δ and the system is under equilibrium as mg = weight $= k\delta$, where k is the spring stiffness, expressed as force per unit length. If now the mass m is given a displacement x in the downward direction and the system is allowed to vibrate, we have the following forces acting on the system: the spring force, $k(x+\delta)$ acting in the upward direction, inertia force mx'' acting in the upward direction and force mg acting in the direction of displacement x downwards. The equation of motion is written taking equilibrium of forces as:

$$
\begin{aligned}
mx'' &= -k(x+\delta) + mg \\
&= -kx - k\delta + mg \\
&= -kx - k\delta + k\delta
\end{aligned}
$$

Or $mx'' + kx = 0$, which is the governing differential equation for a single degree of freedom system. Rewriting the equation of motion as

$x'' + (k/m)\,x = 0$, we have the quantity $(k/m)^{1/2}$ as the natural frequency of the system ω_n .

Energy Method:

In this method the concept of total energy of the system, which is the sum of Kinetic energy (T) and Potential energy(V) , is made use of which remains constant always for any configuration of system while it is vibrating

For a single DOF system of spring and mass, the kinetic energy is given by $(1/2)mx''$ and the potential energy stored in the system is $(1/2)kx^2$. As the total energy of the system remains constant, we have $T+V = 0$ or $d(T+V)/dt = 0$. Differentiating we have the governing differential equation as $mx'' + kx = 0$, and the natural frequency is given by

$\omega_n = (k/m)^{1/2}$.

Rayleigh's Method:

In this method the max kinetic energy of the system is equated to the maximum potential energy. For SHM the max. kinetic energy is at the mean position which is equated to the potential energy. If A is the amplitude of vibration and ω_n is the natural frequency the max. kinetic energy is given by $(1/2)m(\omega_n A)^2$ and max. potential energy is $(1/2)kA^2$. Equating the two equations and simplifying we have again $\omega_n = (k/m)^{1/2}$.

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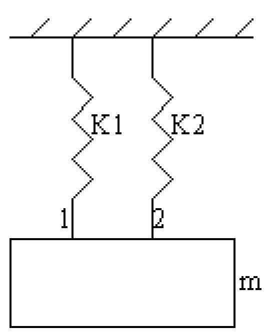
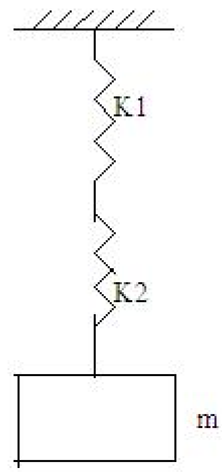
Session 5 date:(7/3/07)

SPRINGS IN SERIES AND SPRINGS IN PARALLEL



a. Springs in parallel



b. Springs in series

Consider figure (b) where the springs are in series. When the mass is subjected to a force 'F', the displacement of mass 'm' is equal to deflections of sprimgs 1 & 2. Hence we can write, the displacement of the equivalent spring as,

$$б = б_1 + б_2$$

Where $б_1$ – deflection of spring 1.and $б_2$ – deflection of spring 2.

Hence we can write

$F/K_e = F/K_1 + F/K_2$, where K_e= equivalent spring stiffness

$1/K_e = 1/K_1 + 1/K_2$

Considering fig (a) where springs are in parallel when the mass is subjected to a force F we have the total spring force equal to sum of individual spring forces.

Hence, we can write the total force in the equivalent spring as

$K_e .x = K_1.x + K_2.x$

Therefore $K_e = K_1 + K_2$

Therefore the equivalent spring stiffness for springs in parallel is equal to

$Ke = K_1 + K_2$

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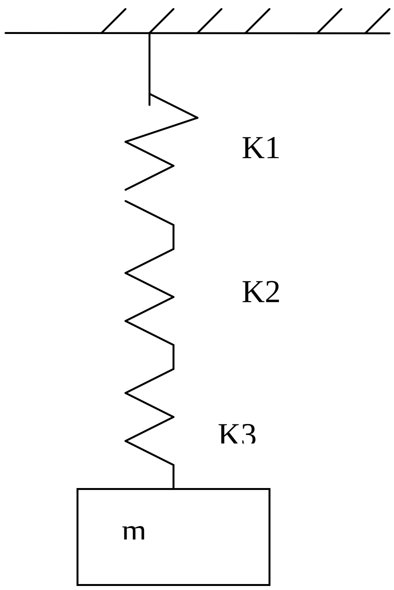
Problem:

1) Obtain an equivalent spring mass system and expression for ω_n. for 3 springs in series and in parallel configuration



i) for series spring combination.

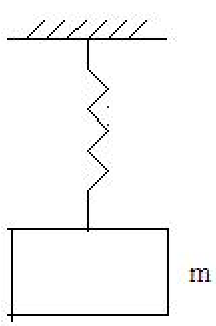
$1/K_e = 1/K_1 + 1/K_2 + 1/K_3$

Therefore $K_e = \dfrac{K_1.K_2.K_3}{K_1K_2 + K_2K_3 + K_1K_3}$

ii) for parallel spring combination.

$K_e = K_1 + K_2 + K_3$

$1/K_e = 1/K_1 + 1/K_2 + 1/K_3$



$K_e = K_1 + K_2 + K_3$

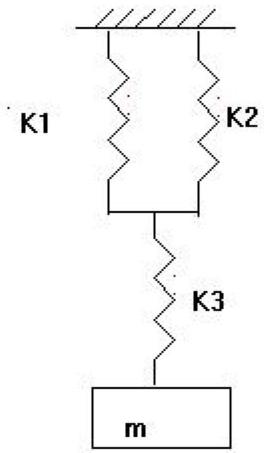
K_e

Natural frequency

$\omega_n = \sqrt{(K_e / m)}$, Therefore $\omega_n = \sqrt{\dfrac{K1 + K2 + K3}{m}}$

Therefore $\omega_n = \sqrt{\dfrac{(1/K1 + 1/K2 + 1/K3)}{m}}$

2). Obtain the natural frequency of the system



Solution:

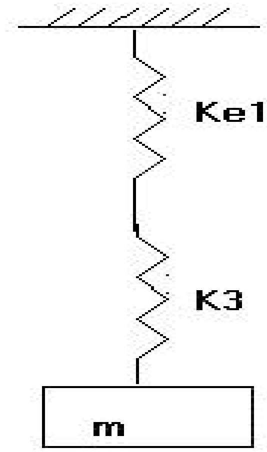
Given m = 109 N

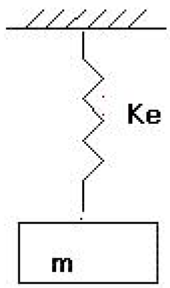
K1 = 10N/mm

K2 = 10N/mm

K3 = 5N/mm

The spring equivalent when parallel springs are added, we have





K_{e1} = K1 + K2
= 20 N/mm

Ke = $\frac{Ke1 + K3}{Ke1 + K3}$

= $\frac{20\ (5)}{25}$

= 4 N/mm

= 4000 N/m

$\omega_n = \sqrt{(K_e / m)}$

Therefore ω_n = 18.97 rad/s

Frequency = fn = $\omega_n / 2\pi$

= 3.012 Hz

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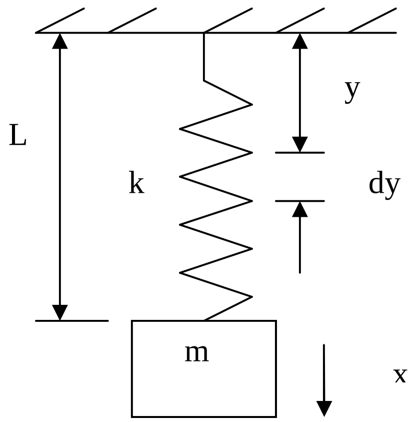
Session 6 date:(9/3/07)

NATURAL FREQUENCY OF A SPRING: Considering mass of spring



Consider a spring mass system as shown in the figure where the mass is displaced by ‘x’. ‘dy’ is a small elemental spring length at a distance of y from the fixed end. ‘L’ be the length of the spring. Let x’ and x” be the velocity and acceleration of mass.

The total K.E of the system is the sum of K.E of the mass ‘m’ and K.E considering the mass of the spring.

The velocity of the spring element at a distance of ‘y’ from the fixed end is x’y/L

We can write the K.E of the spring element ‘dy’ as $(½)(Rho)(dy)(x'y/L)^2$

Where Rho is the mass density

Above expression is of the form $KE = ½ mv^2$.

The K.E for the entire spring considering the mass of the spring becomes

$$\int_0^L (½)(Rho)(dy)(x'y/L)^2$$

$$= (Rho)\, x'^2 / 2l^2 \int_0^L (y^3/3)$$

$$= 1/6\, (Rho)x'^2 L$$

$$= 1/6\, M_s.x'^2 \quad \text{where, } M_s = (Rho)L = \text{Mass spring}$$

Therefore the entire K.E of the system

$K.E = 1/2mx'^2 + 1/6\ M_s.x'^2$

The potential energy of the system

$P.E = ½\ Kx^2$

Therefore the total energy of the system

= K.E + P.E = constant

$1/2mx'^2 + 1/6\ M_s.x'^2 + ½\ Kx^2 = constant$

Differentiating the above expression w.r.t time we get

$Mx'x'' + 1/3\ M_s x'x'' + Kxx' = 0$

$Mx'' + 1/3\ M_s x'' + Kx = 0$

$x''\ \{m + 1/3\ M_s\} + Kx = 0$

From the above expression,

Therefore , $\omega_n = (K/(m+1/3(M_s))^{1/2}$, rad/s

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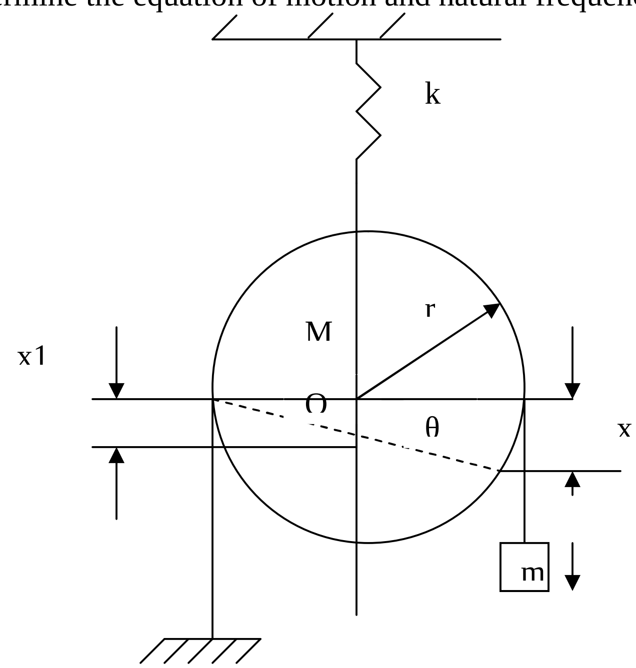
<u>Session 7 date :(16/3/07)</u>

<u>Problems</u>

3) Determine the equation of motion and natural frequency of the system shown



Solution:
It is assumed that:

- The string is inextensible
- The friction between string and disc is neglected.

The disc is given an angular displacement ‘θ’, due to which the mass ‘m’ is displaced by ‘x’, from the figure, we have;

$$x = 2r\,\theta.$$

Also, the vertical displacement of centre ‘O’ is

$X_1 = r\,\theta$

By making use of the energy principle, we have the total energy of the system is constant at any given instant of time.

K.E. System = $1/2mx'^2 + 1/2M\theta'^2 + 1/2I\theta''^2$

$= 1/2\ m(2r\theta')^2 + 1/2M(r\theta')^2 + 1/2(1/2Mr^2)(\theta')^2$

Simplifying

$= r^2\ (\theta')^2(2m+(3/4)M)$

Similarly, P.E. of the system is the strain energy stored in the spring due to displacement of centre x1, i.e, $(1/2)k(x1)^2$, which is $(1/2)k(r^2)(\theta)^2$

According to Energy Method, $\frac{d}{dt}$ (KE+PE) = 0

Differentiating the sum of KE and PE

$(2m+(3/4)M)2\theta''+k\theta = 0$

Or

$\theta''+ k/(4m+(3/2)m) = 0$, which is in the form $\theta''+ \omega_n^2 = 0$

or

$\omega_n^2 = k/(4m+(3/2)m)$, i.e.

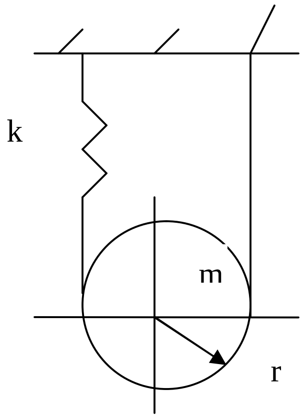
$\omega_n = k/(4m+(3/2)m)^{1/2}$

and the natural frequency inHz, ,

$f_n = (k/(4m+(3/2)m)^{1/2})/2\pi$

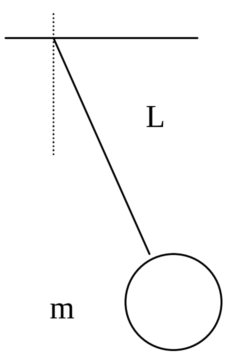
University Problems for practice

1) A homogenous cylinder of mass m and radius r is suspended by a spring and an inextensible cord as shown. Obtain the equation of motion and find the natural frequency of vibration of the cylinder.



Answer: $\theta'' + (8k/3m)\theta = 0$, $f_n = (8k/3m)/2\pi$, Hz.

2) A simple pendulum is as shown in fig. Determine the natural frequency of the system if the mass of the rod m_r is not negligible .



Answer: $\theta'' + ((m + (m_r/2))/((m+(m_r/3))(g/L) = 0$
$f_n = (\sqrt{((m + (m_r/2))/((m+(m_r/3))(g/L))}/2\pi$, Hz

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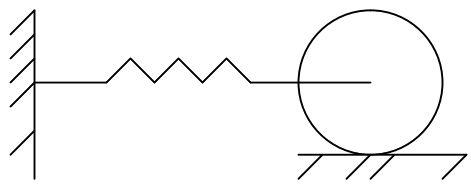
3) A circular cylinder of mass m and mass moment of inertia I is connected by a spring of stiffness k as shown. If it is free to roll without slipping, determine the natural frequency.



Answer: $f_n = (\sqrt{(2k/3m)})/2\pi$, Hz.

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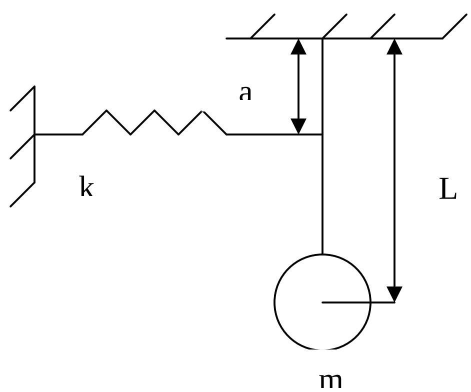
4) The Mass of an uniform rod is negligible compared to the mass attached to it. For small oscillations, calculate the natural frequency of the system.



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FORCED VIBRATIONS

INTRODUCTION

When a mechanical system undergoes free vibrations, an initial force (causing some displacement) is impressed upon the system, and the system is allowed to vibrate under the influence of inherent elastic properties. The system however, comes to rest, depending upon the amount of damping in the system.

In engineering situations, there are instances where in an external energy source causes vibrations continuously acting on the system. Then the system is said to undergo forced vibrations, as it vibrates due to the influence of external energy source. The external energy source may be an externally impressed force or displacement excitation impressed upon the system. The excitation may be periodic, impulsive or random in nature. Periodic excitations may be harmonic or non harmonic but periodic. The amplitude of vibrations remains almost constant. Machine tools, internal combustion engines, air compressors, etc are few examples that undergo forced vibration.

3.2 FORCED VIBRATIONS OF SINGLE DOF SYSTEMS UNDER HARMONIC EXCITATION

Consider a spring mass damper system as shown in Figure 3.1 excited by a sinusoidal forcing function $F=F_0 \sin\varpi t$

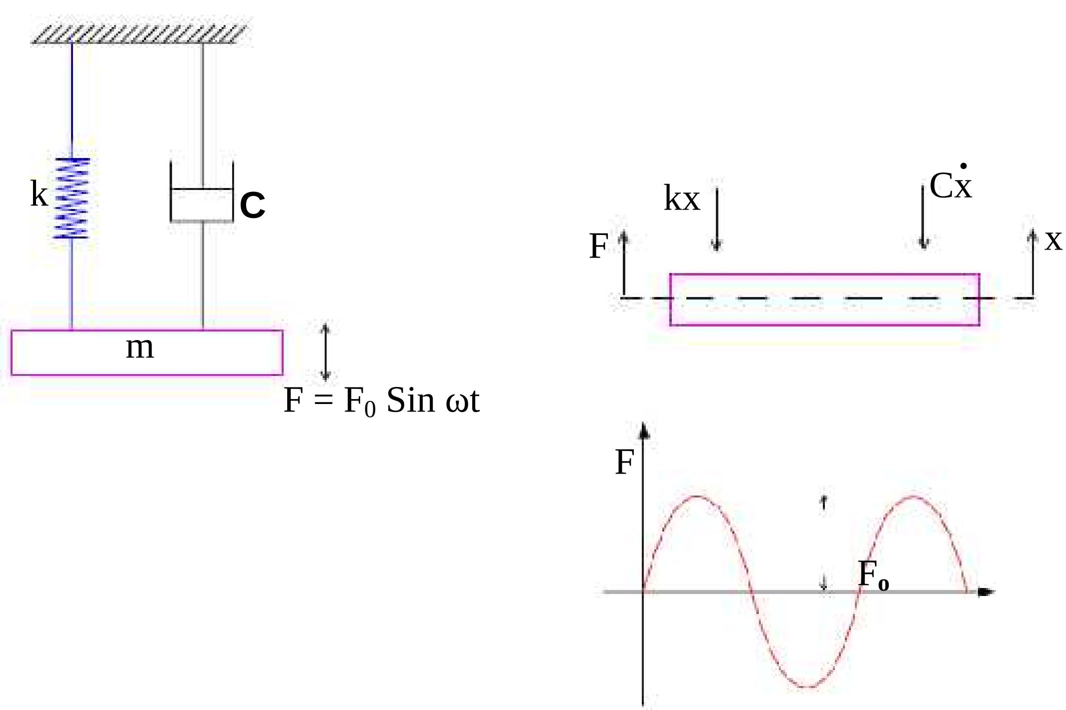

Figure 3.1

Let the force acts vertically upwards as shown in FBD. Then the Governing Differential Equation (GDE) can be written as

$m\ddot{x} = - Kx - C\dot{x} + F$

$m\ddot{x} + C\dot{x} + Kx = F$ --------------- (3.1)

is a linear non homogeneous II order differential equation whose solution is in two parts.

1. Complementary Function or Transient Response

   Consider the homogenous differential equation,

   $m\ddot{x} + C\dot{x} + Kx = 0$ which is incidentally the GDE of a single DOF spring mass damper-system. It has been shown in earlier discussions that for different conditions of damping, the response decays with time. Thus the response is transient in nature and therefore termed as transient response.

   For an under damped system the complementary function or transient response.

   $x_c = X_1 e^{-\zeta\omega_n t} \text{ Sin } (\omega_d t + \phi)$

   $x_c = X_1 e^{-\zeta\omega_n t} (A \text{ Sin } \omega_d t + B \text{ Cos } \omega_d t)$ ------------ (3.2)

2. Particular Integral or Steady State Response

   This response neither builds up nor decays with time. It is steady state harmonic oscillation having frequency equal to that of excitation. It can be determined as follows.

   Consider non-homogenous differential equation

   $m\ddot{x} + C\dot{x} + Kx = F_o \text{ Sin } \omega t$ ----------- (3.3)

The particular integral or steady state response is a steady state oscillation of the same frequency ω as that of external excitation and the displacement vector lags the force vector by some angle.

Let x = X Sin (ωt - ϕ) be the trial solution

X: Amplitude of oscillation

ϕ: Phase of the displacement with respect to the exciting force (angle by which the displacement vector lags the force vector).

∴ Velocity = $\dot{x}$

$\dot{x}$ = ωX. Cos (ωt - ϕ)

= ωX Sin [90 + (ωt - ϕ)]

Acceleration

$\ddot{x} = - \omega^2 X. \text{ Sin } (\omega t - \phi)$, substitute these values in GDE, (equation 3.1)

We get

$-m \omega^2 X \text{ Sin } (\omega t - \phi) + C\omega \text{ Sin } [90 + (\omega t - \phi)]$

$+ KX \text{ Sin } (\omega t - \phi) = F_o \text{ Sin } \omega t$

$m \omega^2 X \text{ Sin } (\omega t - \phi) - C\omega \text{ Sin } [90 + (\omega t - \phi)]$

$- KX \text{ Sin } (\omega t - \phi) + F_o \text{ Sin } \omega t = 0$ ---------- (3.4)