Mechanical Vibrations

Notes

Visvesvaraya Technological University

E-notes

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06ME 64 - MECHANICAL VIBRATIONS

UNIT - 1

Introduction: When an elastic body such as, a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a vibratory motion, due to the elastic or strain energy present in the body. When the body reaches the equilibrium position, the whole of the elastic or stain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The entire KE is again converted into strain energy due to which the body again returns to the equilibrium position. Hence the vibratory motion is repeated indefinitely.

Oscillatory motion is any pattern of motion where the system under observation moves back and forth across some equilibrium position, but dose not necessarily have any particular repeating pattern.

Periodic motion is a specific form of oscillatory motion where the motion pattern repeats itself with a uniform time interval. This uniform time interval is referred to as the *period* and has units of seconds per cycle. The reciprocal of the period is referred to as the *frequency* and has units of cycles per second. This unit of combination has been given a special unit symbol and is referred to as *Hertz* (Hz)

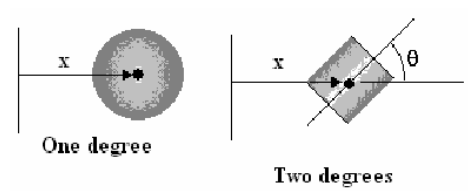
Harmonic motion is a specific form of periodic motion where the motion pattern can be describe by either a sine or cosine. This motion is also sometimes referred to as simple harmonic motion. Because the sine or cosine technically used angles in *radians*, the frequency term expressed in the units radians per seconds (*rad/sec*). This is sometimes referred to as the *circular frequency.* The relationship between the frequency in Hz (cps) and the frequency in *rad/sec* is simply the relationship. 2π *rad/sec*.

Natural frequency is the frequency at which an undamped system will tend to oscillate due to initial conditions in the absence of any external excitation. Because there is no damping, the system will oscillate indefinitely.

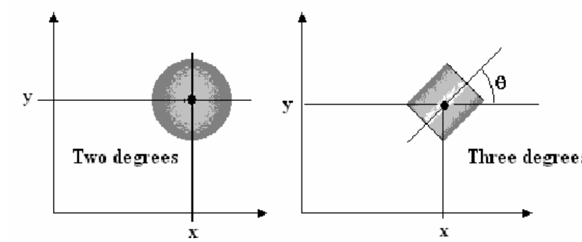
Damped natural frequency is frequency that a damped system will tend to oscillate due to initial conditions in the absence of any external excitation. Because there is damping in the system, the system response will eventually decay to rest.

Resonance is the condition of having an external excitation at the natural frequency of the system. In general, this is undesirable, potentially producing extremely large system response.

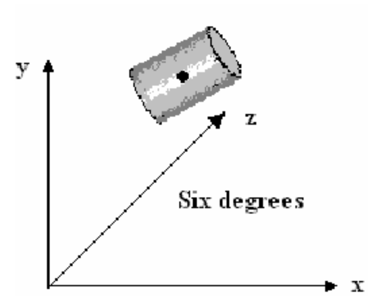
Degrees of freedom: The numbers of degrees of freedom that a body possesses are those necessary to completely define its position and orientation in space. This is useful in several fields of study such as robotics and vibrations. Consider a spherical object that can only be positioned somewhere on the *x* axis.



This needs only one dimension, 'x' to define the position to the centre of gravity so it has one degree of freedom. If the object was a cylinder, we also need an angle 'θ' to define the orientation so it has two degrees of freedom.



Now consider a sphere that can be positioned in Cartesian coordinates anywhere on the z plane. This needs two coordinates 'x' and 'y' to define the position of the centre of gravity so it has two degrees of freedom. A cylinder, however, needs the angle 'θ' also to define its orientation in that plane so it has three degrees of freedom.



In order to completely specify the position and orientation of a cylinder in Cartesian space, we would need three coordinates x, y and z and three angles relative to each angle. This makes six degrees of freedom. A rigid body in space has ($x,y,z,\theta_x\ \theta_y\ \theta_z$).

In the study of free vibrations, we will be constrained to one degree of freedom.

Types of Vibrations:

Free or natural vibrations: A free vibration is one that occurs naturally with no energy being added to the vibrating system. The vibration is started by some input of energy but the vibrations die away with time as the energy is dissipated. In each case, when the body is moved away from the rest position, there is a natural force that tries to return it to its rest position. Free or natural vibrations occur in an elastic system when only the internal restoring forces of the system act upon a body. Since these forces are proportional to the displacement of the body from the equilibrium position, the acceleration of the body is also proportional to the displacement and is always directed towards the equilibrium position, so that the body moves with SHM.

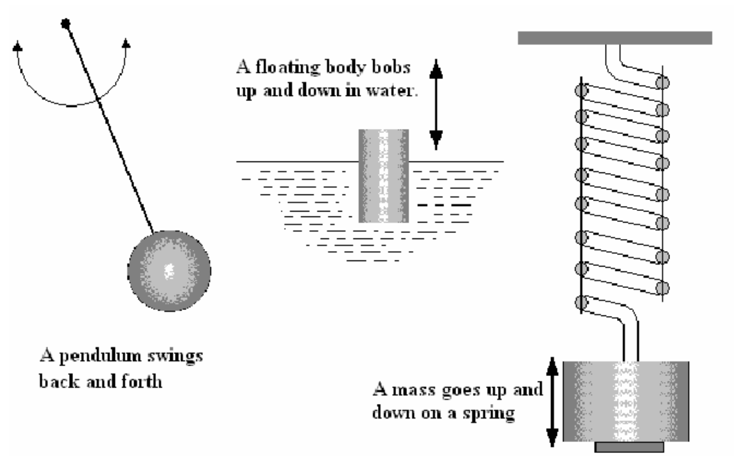


Figure 1. Examples of vibrations with single degree of freedom.

Note that the mass on the spring could be made to swing like a pendulum as well as bouncing up and down and this would be a vibration with two degrees of freedom. The number of degrees of freedom of the system is the number of different modes of vibration which the system may posses.

The motion that all these examples perform is called SIMPLE HARMONIC MOTION (S.H.M.). This motion is characterized by the fact that when the displacement is plotted against time, the resulting graph is basically sinusoidal. Displacement can be linear (e.g. the distance moved by the mass on the spring) or angular (e.g. the angle moved by the simple pendulum). Although we are studying natural vibrations, it will help us understand S.H.M. if we study a forced vibration produced by a mechanism such as the Scotch Yoke.

Simple Harmonic Motion

The wheel revolves at ω radians/sec and the pin forces the yoke to move up and down. The pin slides in the slot and Point *P* on the yoke oscillates up and down as it is constrained to move only in the vertical direction by the hole through which it slides. The motion of point *P* is simple harmonic motion. Point *P* moves up and down so at any moment it has a displacement *x*, velocity *v* and an acceleration *a*.

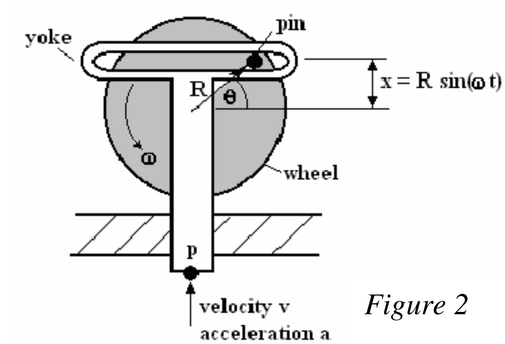
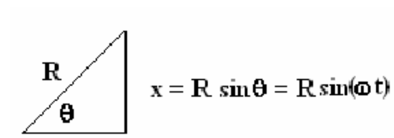


Figure 2

The pin is located at radius *R* from the centre of the wheel. The vertical displacement of the pin from the horizontal centre line at any time is *x*. This is also the displacement of point *P*. The yoke reaches a maximum displacement equal to *R* when the pin is at the top and –*R* when the pin is at the bottom.



This is the amplitude of the oscillation. If the wheel rotates at ω radian/sec then after time *t* seconds the angle rotated is $\theta = \omega t$ radians. From the right angle triangle we find $x = R\,Sin(\omega t)$ and the graph of x - θ is shown on *figure 3a*.

Velocity is the rate of change of distance with time. The plot is also shown on figure 3*a*.

$$v = dx/dt = \omega R\,Cos(\omega t).$$

The maximum velocity or amplitude is ωR and this occurs as the pin passes through the horizontal position and is plus on the way up and minus on the way down. This makes sense since the tangential velocity of a point moving in a circle is $v = \omega R$ and at the horizontal point they are the same thing.

Acceleration is the rate of change of velocity with time. The plot is also shown on *figure 3a*.

$$a = dv/dt = -\omega^2 R\,Sin(-\omega^2 R)$$

The amplitude is $\omega^2 R$ and this is positive at the bottom and minus at the top (when the yoke is about to change direction)

Since $R\,Sin(\omega R) = x$; then substituting x we find $a = -\omega^2 x$

This is the usual definition of S.H.M. The equation tells us that any body that performs sinusoidal motion must have an acceleration that is directly proportional to the displacement and is always directed to the point of zero displacement. The constant of proportionality is ω^2. Any vibrating body that has a motion that can be described in this way must vibrate with S.H.M. and have the same equations for displacement, velocity and acceleration.

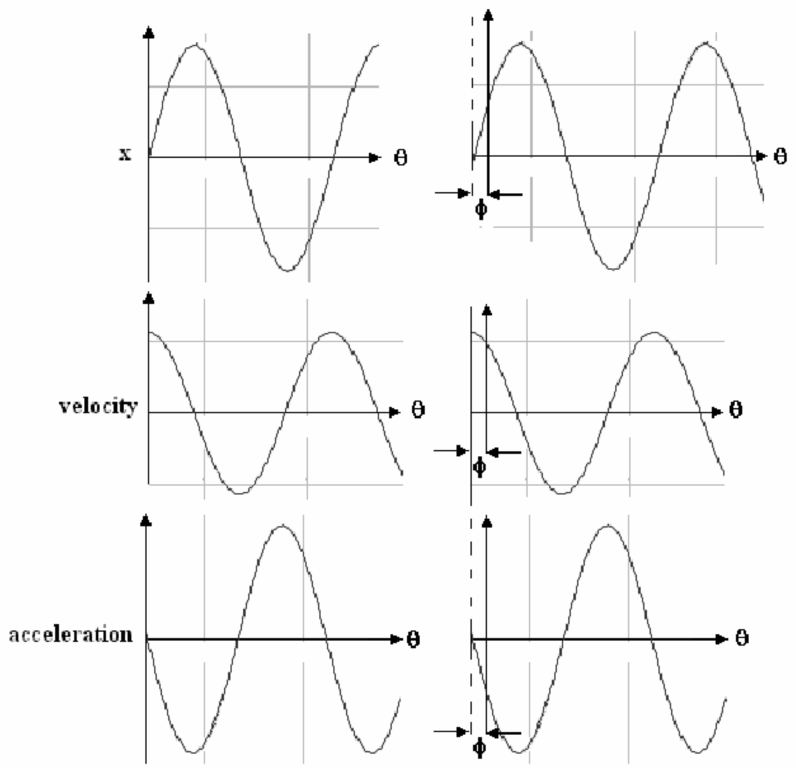


FIGURE 3a *FIGURE 3b*

Angular Frequency, Frequency and Periodic time

ω is the angular velocity of the wheel but in any vibration such as the mass on the spring, it is called the angular frequency as no physical wheel exists.

The frequency of the wheel in revolutions/second is equivalent to the frequency of the vibration. If the wheel rotates at 2 rev/s the time of one revolution is ½ seconds. If the wheel rotates at 5 rev/s the time of one revolution is $^1/_5$ second. If it rotates at f rev/s the time of one revolution is $^1/_f$. This formula is important and gives the periodic time.

Periodic Time T = time needed to perform one cycle.
f is the frequency or number of cycles per second.

It follows that: $T = {}^1/_f$ and $f = {}^1/_T$

Each cycle of an oscillation is equivalent to one rotation of the wheel and 1 revolution is an angle of 2π radians.

When $\theta = 2\pi$ and $t = T$.

It follows that since $\theta = \omega t$; then $2\pi = \omega T$

Rearrange and $\theta = {}^{2\pi}/_T$. Substituting $T = {}^1/_f$, then $\omega = 2\pi f$

Equations of S.H.M.

Consider the three equations derived earlier.

Displacement $x = R\ Sin(\omega t)$.

Velocity $v = dx/dt = \omega R\ Cos(\omega t)$ and Acceleration $a = dv/dt = -\omega^2 R\ Sin(\omega t)$

The plots of *x, v* and *a* against angle θ are shown on figure 3*a*. In the analysis so far made, we measured angle θ from the horizontal position and arbitrarily decided that the time was zero at this point.

Suppose we start the timing after the angle has reached a value of ϕ from this point. In these cases, ϕ is called the phase angle. The resulting equations for displacement, velocity and acceleration are then as follows.

Displacement $x = R\ Sin(\omega t + \phi)$.

Velocity $v = dx/dt = \omega R\ Cos(\omega t + \phi)$.

Acceleration $a = dv/dt = -\omega^2 R\ Sin((\omega t + \phi)$.

The plots of *x, v* and *a* are the same but the vertical axis is displaced by ϕ as shown on figure 3*b*. A point to note on figure 3*a and* 3*b* is that the velocity graph is shifted ¼ cycle (90^o) to the left and the acceleration graph is shifted a further ¼ cycle making it ½ cycle out of phase with *x*.

Forced vibrations: When the body vibrates under the influence of external force, then the body is said to be under forced vibrations. The external force, applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.
(Note: When the frequency of external force is same as that of the natural vibrations, resonance takes place)

Damped vibrations: When there is a reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration. This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistance to the motion.

Types of free vibrations:

Linear / Longitudinal vibrations: When the disc is displaced vertically downwards by an external force and released as shown in the *figure 4*, all the particles of the rod and disc move parallel to the axis of shaft. The rod is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the rod. The vibration occurs is know as *Linear/Longitudinal vibrations*.

Transverse vibrations: When the rod is displaced in the transverse direction by an external force and released as shown in the *figure 5*, all the particles of rod and disc move approximately perpendicular to the axis of the rod. The shaft is straight and bends alternately inducing bending stresses in the rod. The vibration occurs is know as *transverse vibrations*.

Torsional vibrations: When the rod is twisted about its axis by an external force and released as shown in the *figure 6*, all the particles of the rod and disc move in a circle about the axis of the rod. The rod is subjected to twist and torsional shear stress is induced. The vibration occurs is known as *torsional vibrations.*

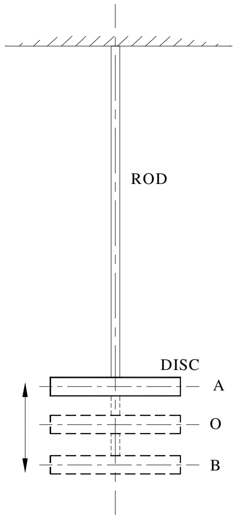


FIGURE 4

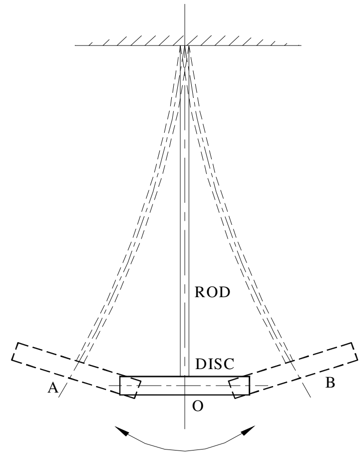


FIGURE 5

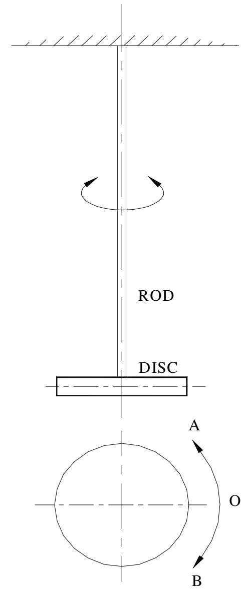
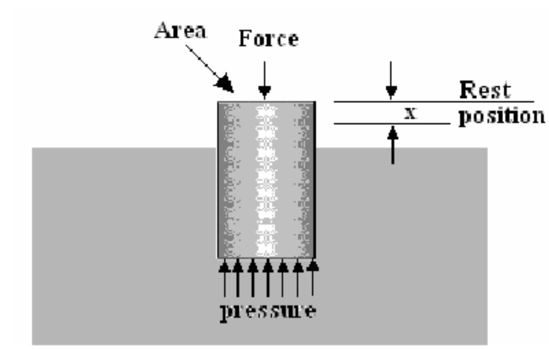


FIGURE 6

***Oscillation of a floating body*:**

You may have observed that some bodies floating in water bob up and down. This is another example of simple harmonic motion and the restoring force in this case is **buoyancy**.
Consider a floating body of mass M kg. Initially it is at rest and all the forces acting on it add up to zero. Suppose a force F is applied to the top to push it down a distance x. The applied force F must overcome this buoyancy force and also overcome the inertia of the body.



Buoyancy force:
The pressure on the bottom increases by $\Delta p = \rho\, g\, x$.
The buoyancy force pushing it up on the bottom is F_b and this increases by $\Delta p\, A$.
Substitute for Δp and $F_b = \rho\, g\, x\, A$

Inertia force:
The inertia force acting on the body is $F_i = M\, a$

Balance of forces:
The applied force must be $F = F_i + F_b$ -this must be zero if the vibration is free.

$$0 = Ma + \rho\, g\, x\, A$$

$$a = -\frac{\rho A g}{M}x$$

This shows that the acceleration is directly proportional to displacement and is always directed towards the rest position so the motion must be simple harmonic and the constant of proportionality must be the angular frequency squared.

$$\omega^2 = \frac{\rho A g}{M}$$

$$\omega = \sqrt{\frac{\rho A g}{M}}$$

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\rho A g}{M}}$$

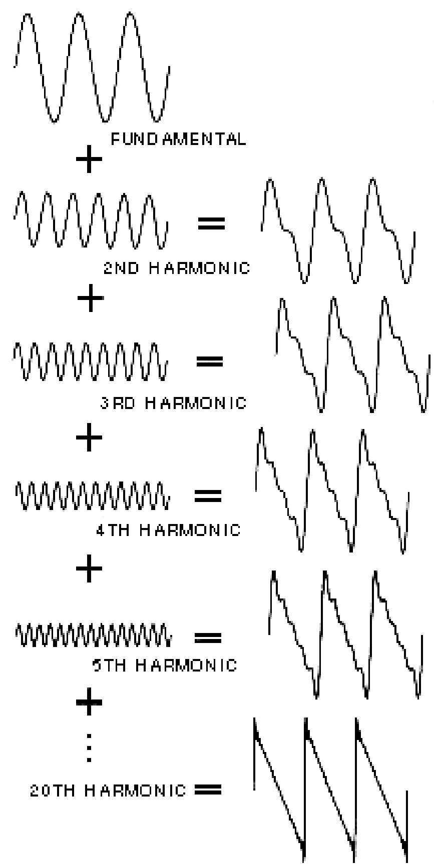
Example: A cylindrical rod is 80 mm diameter and has a mass of 5 kg. It floats vertically in water of density 1036 kg/m3. Calculate the frequency at which it bobs up and down. (Ans. 0.508 Hz)

Principal of super position:

The principal of super position means that, when TWO or more waves meet, the wave can be added or subtracted.

Two waveforms combine in a manner, which simply adds their respective Amplitudes linearly at every point in time. Thus, a complex SPECTRUM can be built by mixing together different Waves of various amplitudes.

The principle of superposition may be applied to waves whenever two (or more) waves traveling through the same medium at the same time. The waves pass through each other without being disturbed. The net displacement of the medium at any point in space or time, is simply the sum of the individual wave dispacements.



General equation of physical systems is:

$m\ddot{x} + c\dot{x} + kx = F(t)$ - This equation is for a linear system, the inertia, damping and spring force are linear function $\ddot{x}, \dot{x}$ *and* x respectively. This is not true case of non-linear systems.

$m\ddot{x} + \phi(\dot{x}) + f(x) = F(t)$ - Damping and spring force are not linear functions of $\dot{x}$ *and* x

Mathematically for linear systems, if x_1 is a solution of;

$$m\ddot{x} + c\dot{x} + kx = F_1(t)$$

and x_2 is a solution of;

$$m\ddot{x} + c\dot{x} + kx = F_2(t)$$

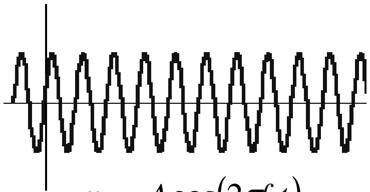
then $(x_1 + x_2)$ is a solution of;

$$m\ddot{x} + c\dot{x} + kx = F_1(t) + F_2(t)$$

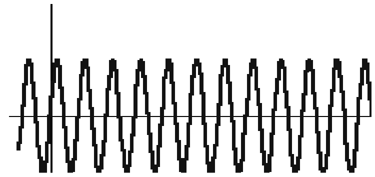
Law of superposition does not hold good for non-linear systems.

If more than one wave is traveling through the medium: The resulting net wave is given by the *Superposition Principle given by the sum of the individual waveforms"*

Beats: When two harmonic motions occur with the same amplitude '*A*' at different frequency is added together a phenomenon called "beating" occurs.



$$y_1 = A\cos(2\pi f_1 t)$$



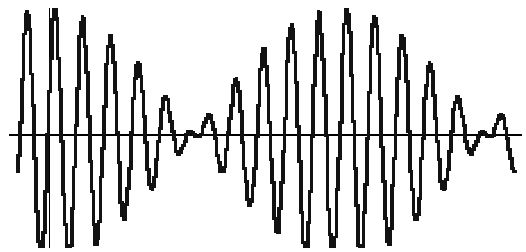
$$y_2 = A\cos(2\pi f_2 t)$$

The resulting motion is:

$$y = (y_1 + y_2) = A[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)]$$

with trigonometric manipulation, the above equation can be written as:

$$y = 2A\cos 2\pi \frac{f_1 - f_2}{2} t \times \cos 2\pi \frac{f_1 + f_2}{2} t$$



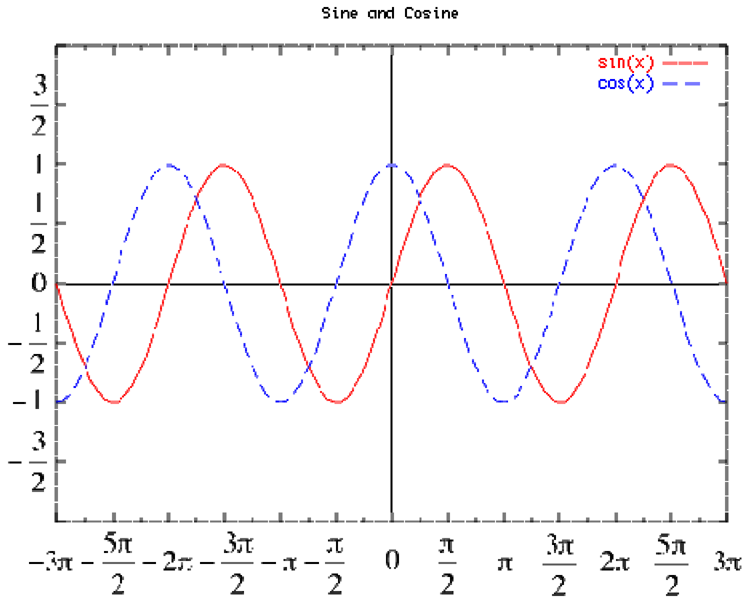
The resultant waveform can be thought of as a wave with frequency $f_{ave} = (f_1 + f_2)/2$ which is constrained by an envelope with a frequency of $f_b = |f_1 - f_2|$. The envelope frequency is called the beat frequency. The reason for the name is apparent if you listen to the phenomenon using sound waves.

(Beats are often used to tune instruments. The desired frequency is compared to the frequency of the instrument. If a beat frequency is heard the instrument is "out of tune". The higher the beat frequency the more "out of tune" the instrument is.)

Fourier series: decomposes any *periodic function* or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions, namely *sines and cosines* (or complex exponentials).
Fourier series were introduced by Joseph Fourier (1768–1830) for the purpose of solving the *heat equation* in a metal plate.

The Fourier series has many such applications in *electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, thin-walled shell theory,etc*.

Sin & Cos functions



J. Fourier, developed a periodic function in terms of series of Sines and Cosines. The vibration results obtained experimentally can be analysed analytically.
If ***x(t)*** is a periodic function with period *T*, the Fourier Series can be written as:

$$x(t) = \frac{a_o}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \ldots\ldots\ldots\ldots$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \ldots\ldots\ldots\ldots$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

Where $\omega = \frac{2\pi}{T}$ is the fundamental frequency and

$a_0, a_1, a_2, a_3 \ldots\ldots\ldots b_1, b_2, b_3 \ldots\ldots\ldots$ are constant coefficients.

The term $(a_1 \cos \omega t + b_1 \sin \omega t)$ is called the Fundamental or First Harmonic.

The term $(a_2 \cos \omega t + b_2 \sin \omega t)$ is called the second Harmonic and so on.

Example 1.

The displacement of a body performing simple harmonic motion is described by the following equation

$x = A\ Sin(\omega t + \phi)$ where A = amplitude, ω = natural frequency and ϕ = phase angle.

Given A = 20 mm, ω = 50 rad/s and $\phi = \pi/8$ radian, calculate the following.

i. The frequency. ii. The periodic time.

iii. The displacement, velocity and acceleration - when $t = T/4$.

Solution:

The frequency. $f = \omega/2\pi = 50/2\pi = 7.96$ Hz.

The periodic time. $T = 1/f = 0.126$ s

Time t. $t = T/4 = 0.0314$ sec

Equation for displacement (x) at t = 0.0314 sec

$$Displacement = x = A\,Sin(\omega t + \phi)$$
$$= 20\,Sin\left(50 \times 0.0314 + \frac{\pi}{8}\right)$$
$$= 20\,Sin 1.963 = 0.685\,mm$$

$$Velocity = v = A\omega Cos(\omega t + \phi)$$
$$= 20 \times 50 \times Cos(1.963)$$
$$= 999\,mm/s$$

$$Acceleration = a = -A\omega^2\,Sin(\omega t + \phi)$$
$$= -20 \times 50^2 \times Sin(1.963)$$
$$= -1712\,mm/s^2$$

Refer PPT – for more Problems

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06ME 64 - MECHANICAL VIBRATIONS

UNIT - 2

Undamped Free Vibrationis:

NATURAL FREQUENCY OF FREE LONGITUDINAL VIBRATION

Equilibrium Method: Consider a body of mass '*m*' suspended from a spring of negligible mass as shown in the *figure 4*.

Let m = Mass of the body

W = Weight of the body $= mg$

K = Stiffness of the spring

δ = Static deflection of the spring due to '*W*'

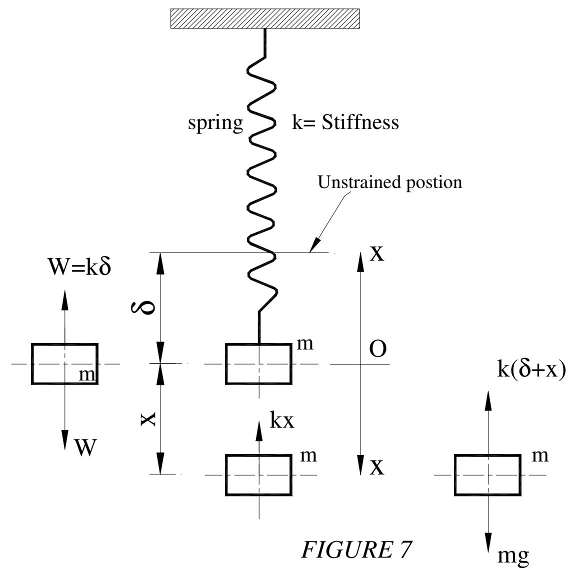


FIGURE 7

By applying an external force, assume the body is displaced vertically by a distance '*x*', from the equilibrium position. On the release of external force, the unbalanced forces and acceleration imparted to the body are related by Newton Second Law of motion.

$\therefore$ The restoring force = $F = - k \times x$

(-ve sign indicates, the restoring force 'k.x' is opposite to the direction of the displacement 'x')

By Newton's Law; $F = m \times a$

$$\therefore\ F = -k\,x = m\frac{d^2x}{dt^2}$$

$\therefore$ The differential equation of motion, if a body of mass '*m*' is acted upon by a restoring force '*k*' per unit displacement from the equilibrium position is;

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad - \textit{This equation represents SHM}$$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \omega^2 = \frac{k}{m} \quad - \textit{for SHM}$$

The natural period of vibration is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ *Sec*

The natural frequency of vibration is $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ *cycles* / sec

From the *figure 7*; when the spring is strained by an amount of 'δ due to the weight $W = mg$

$$\delta k = mg$$

Hence $$\frac{k}{m} = \frac{g}{\delta}$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad Hz \;\; or \;\; cps$$

Energy method: The equation of motion of a conservative system may be established from energy considerations. If a conservative system set in motion, the mechanical energy in the system is partially kinetic and partially potential. The *KE* is due to the velocity of mass and the *PE* is due to the stain energy of the spring by virtue of its deformation.

Since the system is conservative; and no energy is transmitted to the system and from the system in the free vibrations, the sum of *PE* and *KE* is constant. Both velocity of the mass and deformation of spring are cyclic. Thus, therefore be constant interchange of energy between the mass and the spring.

(KE is maximum, when PE is minimum and PE is maximum, when KE is minimum - so system goes through cyclic motion)

$$KE + PE = Constant$$

$$\frac{d}{dt}\left[KE + PE\right] = 0 \qquad -(1)$$

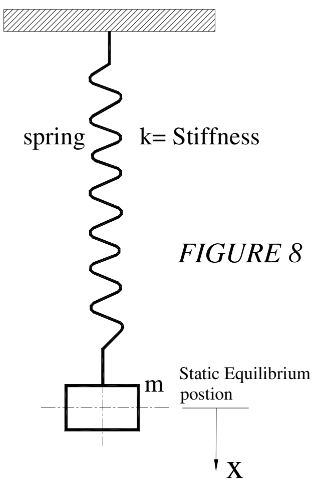


FIGURE 8

We have $KE = \frac{1}{2} m v^2 = \frac{1}{2} m\left(\frac{dx}{dt}\right)^2$ -(2)

Potential energy due to the displacement is equal to the strain energy in the spring, minus the *PE* change in the elevation of the mass.

$$\therefore PE = \int_0^x (Total \;\; spring \;\; forece) dx - mg\, dx$$

$$= \int_0^x (mg + kx - mg) dx = \frac{1}{2} kx^2 \qquad -(3)$$

$$\left[PE = \left(\frac{0 + kx}{2}\right) x = \frac{1}{2} kx^2\right]$$

Equation (1) becomes

$$\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2\right] = 0$$

$$\left(m\frac{d^2x}{dt^2} + kx\right)\frac{dx}{dt} = 0$$

$$Either \left(m\frac{d^2x}{dt^2} + kx\right) = 0 \qquad OR \qquad \frac{dx}{dt} = 0$$

$$But \;\; velocity \;\; \frac{dx}{dt} \;\; can \;\; be \;\; zero \;\; for \;\; all \;\; values \;\; of \;\; time.$$

$$\therefore m\frac{d^2x}{dt^2} + kx = 0 \qquad [m\ddot{x} + kx = 0]$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad - Equation \;\; represents \;\; SHM$$

$$\therefore Time\ \ period = T = 2\pi\sqrt{\frac{m}{k}}\ \sec \qquad and$$

$$Natural\ \ frequency\ of\ \ vibration = f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\ cycles/\sec$$

(The natural frequency is inherent in the system. It is the function of the system parameters 'k' and 'm' and it is independent of the amplitude of oscillation or the manner in which the system is set into motion.)

Rayleigh's Method: The concept is an extension of energy method. We know, there is a constant interchange of energy between the *PE* of the spring and *KE* of the mass, when the system executes cyclic motion. At the static equilibrium position, the *KE* is maximum and *PE* is zero; similarly when the mass reached maximum displacement (amplitude of oscillation), the *PE* is maximum and *KE* is zero (velocity is zero). But due to conservation of energy total energy remains constant.

Assuming the motion executed by the vibration to be simple harmonic, then;

$$x = A\ Sin\omega t$$

$x =$ displacement of the body from the mean position after time 't' sec and
A = Maximum displacement from the mean position

$$\dot{x} = A\,Sin\,\omega t$$

At mean position, $t = 0$; Velocity is maximum

$$\therefore\ v_{\max} = \left(\frac{dx}{dt}\right)_{\max} = \dot{x}_{\max} = \omega A$$

$$\therefore Maximum\ K.\ E = \frac{1}{2} m\,\omega^2 A^2$$

$$Maximum\quad P.E = \frac{1}{2} k\ x_{\max}^2 \qquad\qquad x_{\max} = A$$

$$\therefore Maximum\ \ P.E = \frac{1}{2} kA^2$$

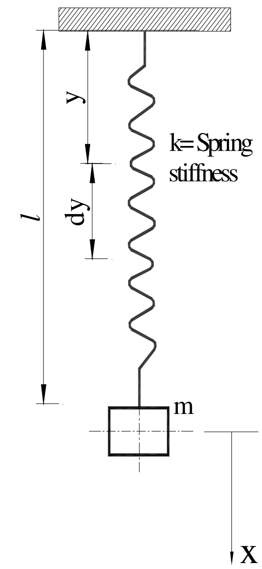
$$We\ know \qquad (KE)_{\max} = (PE)_{\max}$$

$$m\ \omega^2 = k$$

$$\omega = \left(\frac{k}{m}\right)^{1/2}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad\qquad \left[\begin{array}{l} \because \delta\ k = m\ g \\ \Rightarrow \dfrac{k}{m} = \dfrac{g}{\delta} \end{array}\right]$$

1. Determine the natural frequency of the spring-mass system, taking mass of the spring into account.



Let l = Length of the spring under equilibrium condition

ρ = Mass/unit length of the spring

m_s = Mass of the spring = $\rho \times l$

Consider an elemental length of 'dy' of the spring at a distance 'y' from support.

$\therefore$ Mass of the element = ρ dy

At any instant, the mass 'm' is displaced by a

$$\therefore P\,E = \frac{1}{2} k\, x^2$$

K E of the system at this instant, is the sum of $(KE)_{mass}$ and $(KE)_{spring}$

$$\therefore K\,E = \frac{1}{2} m\, \dot{x}^2 + \int_0^l \frac{1}{2}(\rho\, dy)\left(\frac{y}{l}\dot{x}\right)^2$$

$$= \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2}\rho \frac{\dot{x}^2}{l^2}\int_0^l y^2\, dy$$

$$= \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2}\rho \frac{\dot{x}^2}{l^2}\frac{l^3}{3}$$

$$= \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2}\dot{x}^2 \frac{m_s}{3} = \frac{1}{2}\left[m + \frac{m_s}{3}\right]\dot{x}^2$$

$$\frac{1}{2} k\, x^2 + \frac{1}{2}\left(m + \frac{m_s}{3}\right)\dot{x}^2 = 0$$

Differentiating with respect to 't' ; $\frac{d}{dt}(PE + KE) = 0$

$$k\, x\, \dot{x} + \left(m + \frac{m_s}{3}\right)\dot{x}\,\ddot{x} = 0 \qquad \text{– Differential equation}$$

$$\ddot{x} + \frac{k}{\left(m + \frac{m_s}{3}\right)} x = 0$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{k}{\left(m + \frac{m_s}{3}\right)}}\ \text{cps} \qquad [\rho\, l = m_s]$$

OR

$$\therefore f_n = \sqrt{\frac{k}{\left(m + \frac{m_s}{3}\right)}}\ \text{radians/sec}$$

We know that PE + KE = Constant