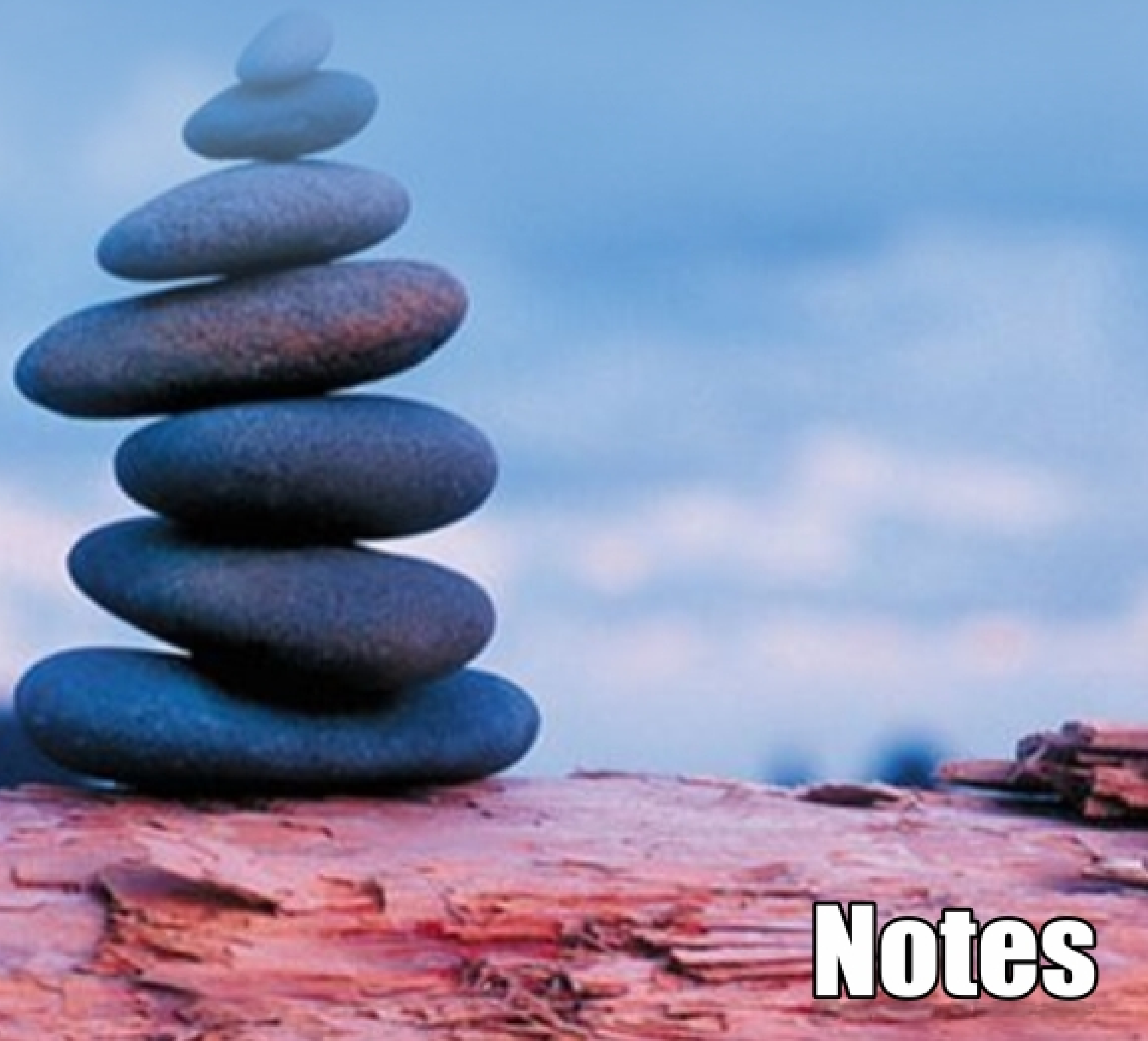
Discrete Mathematics



Notes

The objectives of Discrete Mathematical Structures are:

- To introduce a number of Discrete Mathematical Structures (DMS) found to be serving as tools even today in the development of theoretical computer science.
- Course focuses on of how Discrete Structures actually helped computer engineers to solve problems occurred in the development of programming languages.
- Also, course highlights the importance of discrete structures towards simulation of a problem in computer science and engineering.
- Introduction of a number of case studies involving problems of Computer Technology.

Outcomes of this course are:

- A complete knowledge on various discrete structures available in literature.
- Realization of some satisfaction of having learnt that discrete structures are indeed useful in computer science and engineering and thereby concluding that no mistake has been done in studying this course.
- Gaining of some confidence on how to deal with problems which may arrive in computer science and engineering in near future.
- Above all, students who studied this course are found to be better equipped in a relative sense as far as preparation for entrance examinations involving placement opportunities.

What is Discrete Mathematics then?

- Mathematics is broadly divided into two parts; (i) the continuous mathematics and (ii) the discrete mathematics depending upon the presence or absence of the limiting processes.
- In the case of continuum Mathematics, there do exists some relationship / linkage between various topics whereas Discrete Mathematics is concerned with study of distinct, or different, or un-related topics of mathematics curriculum; it embraces

several topical areas of mathematics some of which go back to early stages of mathematical development while others are more recent additions to the discipline. The present course restricts only to introducing discrete structures which are being used as tools in theoretical computer science.

- A course on Discrete mathematics includes a number of topics such as study of sets, functions and relations, matrix theory, algebra, Combinatorial principles and discrete probability, graph theory, finite differences and recurrence relations, formal logic and predicate calculus, proof techniques - mathematical induction, algorithmic thinking, Matrices, Primes, factorization, greatest common divisor, residues and application to cryptology, Boolean algebra; Permutations, combinations and partitions; Recurrence relations and generating functions; Introduction to error-correcting codes; Formal languages and grammars, finite state machines. linear programming etc. Also, few computer science subjects such as finite automata languages, data structures, logic design, algorithms and analysis were also viewed as a part of this course.
- Because of the diversity of the topics, it is perhaps preferable to treat Discrete Mathematics, simply as Mathematics that is necessary for decision making in non-continuous situations. For these reasons, we advise students of CSE / ISE / MCA, TE (Telecommunication Engineering) to study this course, as they needs to know the procedure of communicating with a computer may be either as a designer, programmer, or, at least a user.
- Of course, in today's situation, this is true for all, although we do not teach to students of other branches of engineering. In some autonomous engineering colleges, DMS is being offered s an elective. Considering these view points, you are informed to undertake a course on Discrete Mathematical Structures so that you will be able to function as informed citizens of an increasingly technological society.
- Also, Discrete Mathematics affords students, a new opportunity to experience success and enjoyment in Mathematics classes. If you have encountered numerous difficulties with computation and the complexities of Mathematics in the past, then may I say that this course is soft and a study requires very few formal skills as prerequisites.

- In case if you are discouraged by the routine aspects of learning Mathematics, Discrete Structures provides you a unique opportunity to learn Mathematics in a much different way than the one employed by your teachers previously. Above all, Discrete Mathematics is vital, exciting, and no doubt is useful otherwise you would not have been suggested to register for this course.
- Further, Discrete Mathematics course serves as a gateway for a number of subjects in computer science and engineering. With these motivations, here, we initiate a detailed discussion on some of the topics: These include Basic set theory, Counting techniques, Formal Logic and Predicate calculus, Relations and functions in CSE, Order relations, Groups and Coding etc.
- Before, continuing, let me mention the difference between Discrete Mathematics and other Mathematics; consider a bag of apples and a piece of wire. In the former, the apples sit apart discretely from each other while in the latter, the points on a wire spread themselves continuously from one end to the other.
- Thus, the numbers 0, 1, 2, 3, . . . are sufficient to handle DMS, where as a real variable taking values continuously over a range of values is required to deal with continuum Mathematics. Hence,
- **Discrete Mathematics + Limiting Processes = Continuum Mathematics.**
- **Prescribed text book:**
- Discrete and Combinatorial Mathematics by R. P. Grimaldi, PHI publications, 5^{th} edition (2004).
- **Reference Books:**
- Discrete Mathematical Structures by Kenneth Rosen, Tata McGraw Hill Publications
- Discrete Mathematical Structures by Kolman, Busby and Ross, PHI publications

Basic set theory

A set is a well defined collection of well defined distinct objects. A set is usually denoted by using upper case letters like A, B, G, T, X etc. and arbitrary elements of the set are denoted using lower case letters such as a, b, g, v etc.

Universal set: The set of all objects under some investigation is called as universe or universal set, denoted by the symbol U.

Consider a set A. Let x be an element of A. This we denote symbolically by $x \in A$. On the other hand, if y is not an element of the set, the same is written as $y \notin A$. Thus, it is clear that with respect to a set A, and an element of the universal set U, there are only two types of relationship possible; (i) the element x under question is a member of the set A or the element x need not be a member of A.

This situation may well be described by using binary numbers 0 and 1. We set $x:1$ to indicate that the element in question is a member of the set A. The notation $x:0$ means that the element under study is not a member of the set A. There are a number of ways of do this task. (i) Writing the elements of a set within the braces. For example, consider A = {dog, apple, dead body, 5, Dr. Abdul Kalam, rose}. Certainly, A qualifies as a set.

(ii) A set may be explained by means of a statement where elements satisfying some conditions. Consider $V = \{x \mid x$ is a Engineering College Affiliated to VTU, Belgaum$\}$

(iii) Definition of a statement may be given by means of a statement like Z denotes the set of all integers. Thus, Z = {. . . -2, -1, 0, 1, 2, 3,. . }.

A null set is a one not having any elements at all. It is denoted by the symbol { } or as .
Give few examples of null set or empty set.

Compliment of a set: Let ***A*** be a set. The compliment of A is defined as a set containing elements of the universe but not the elements of the set A. Thus, $\overline{A} = \{x \in U \mid x \notin A\}$

Subset of a set: Let A and B be two sets. We say that A is a subset of B whenever B contains all the elements of A or equivalently, each element of the set A is a member of B. This is denoted by the symbol $A \subseteq B$. In a construction of a subset, we have the option of including the null set as well as, the set itself.

Power set of a set: Let X be a set, then collection of all subsets of X is called as power set of X, denoted by P(X). Thus, $P(X) = \{A \mid A \subseteq X\}$. For example, if X = {a, b, c}, then power set of X is given as $P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. In general if A set X has n elements, then its power set will have 2^n elements. This is because, during the construction of a subset of X using the n – elements of X, we have a choice of either including an element of X in the subset or excluding the same element in the subset. Thus, each of the n – elements of X has exactly 2 choices, therefore the total number of choices will turn out to be 2^n.

Proper subset of a Set: Let A and B be two sets. One says A is a proper subset of B if B contains at least one element that is not in A. This is denoted by $A \subset B$.

Note: Difference between proper subset and a subset is the following:

- In a subset of X, we can find both the null set and the set X itself.
- In the case of a proper subset of X, we can include the null set but not the set X.

Example of a Discrete Mathematical Structure

Consider the universal set, U. Let $X = \{A, B, C\}$ be a collection of subsets of U. Consider the set operator “subset defined on $X = \{A, B, C\}$. We have **1.** $A \subseteq A$ and $\emptyset \subseteq A$

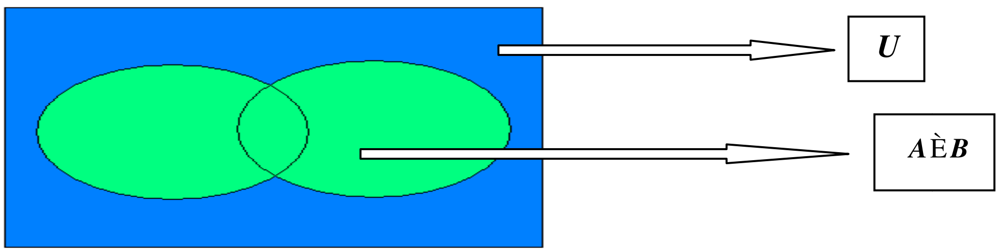
2. If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$, **3.** If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$. Thus, we can claim that $(X, \subseteq)$ an example of a discrete structure.

Equal Sets: Let *A* and *B* be two sets. We say that A = *B* whenever both *A* and *B* have exactly same elements. Equivalently if $A \subseteq B$ and $B \subseteq A$. .

Equivalent sets: Let A and B be two sets. We say A is equivalent to B if A and B have same number of elements. This is denoted by $A \prec B$. The number of elements in a set is called **cardinal number** of the set. This is denoted by $|A|$.

Discussion on Set operations

Union Operator: Consider a universal set Let A, B, C be any three sets in U. Then union of A and B is defined as $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$ The same can be explained by using Venn diagram and by employing a membership table.



The functioning of union operator may also be explained using membership table and Venn diagram. The membership table is given below.

Here, we set up $x : 1$ to mean $x \in A, B$ and **Here, we set up $x : 0$ to mean that $x \notin A, B$**.

A	B	$A \cup B$
1	**1**	**1**
1	**0**	**1**
0	**1**	**1**
0	**0**	**0**

From the membership table, it is clear that $x \notin A \cup B$ only when $x \notin A$ and $x \notin B$. For all other instances, $x \in A \cup B$.

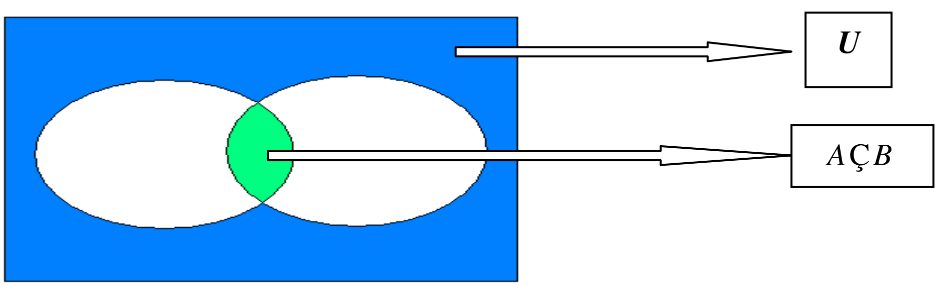
Properties with respect to Union Operator:

1. $A \cup A = A$ (idempotent law)
2. $A \cup B = B \cup A$ (Commutative law)
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
4. $A \cup \phi = \phi \cup A = A$ (Identity law)
5. $A \cup U = U \cup A = U$ (Universal law)
6. If $A \subseteq B$, then $A \cup B = B$

Thus, we have another discrete structure, namely, $(A, B, C, \cup)$ where ***A, B,*** and ***C*** are all subsets of the universal set, ***U***. Before, continuing what is a discrete structure?

A set together with an operation and objects of the set satisfying some properties is called as discrete structure.

Intersection Operator: Consider a universal set Let ***A, B, C*** be any three sets in ***U***. Then Intersection of A and B is defined as $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$. The same can be explained by using Venn diagram and by employing a membership table.



The same well be explained by using membership table

A	B	$A \cap B$
1	**1**	**1**
1	**0**	**0**
0	**1**	**0**
0	**0**	**0**

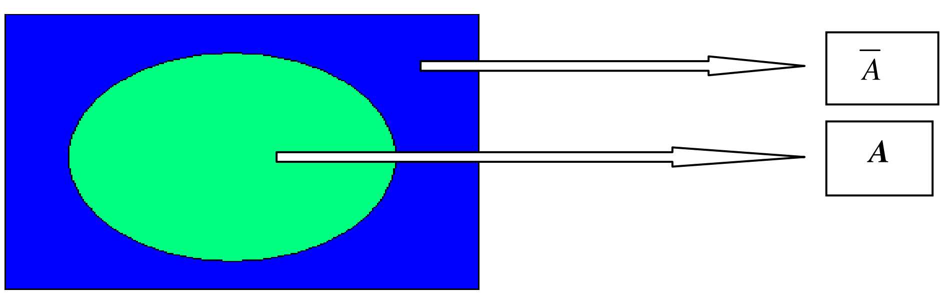
Note: Here, $x \in A \cap B$ only when $x \in A$ and $x \in B$. For other instances, $x \notin A \cap B$. It may be clear that the two operators union and inter-section have just contrasting characteristics. In view of this, these operators are called as dual operators.

Properties with respect to Intersection Operator:

1. $A \cap A = A$ (idempotent law)
2. $A \cap B = B \cap A$ (Commutative law)
3. $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)
4. $A \cap U = U \cap A = A$ (Identity law)
5. $A \cap \phi = \phi \cap A = \phi$ (Universal law)
6. If $A \subseteq B$, then $A \cap B = A$

Thus, we have generated a discrete structure, namely, $(A, B, C, \cap)$ where ***A, B***, and ***C*** are all subsets of the universal set***, U***.

Compliment Operator: Consider a universal set ***U***. Let ***A*** be a subset of an universal set, then the compliment of the set ***A*** is defined and described as $\overline{A} = \{x \in U \mid x \notin A\}$. The Venn diagram and membership table for this operation are:



The membership table is shown below:

A	$\overline{A}$
1	**0**
0	**1**

Properties with respect to compliment Operator:

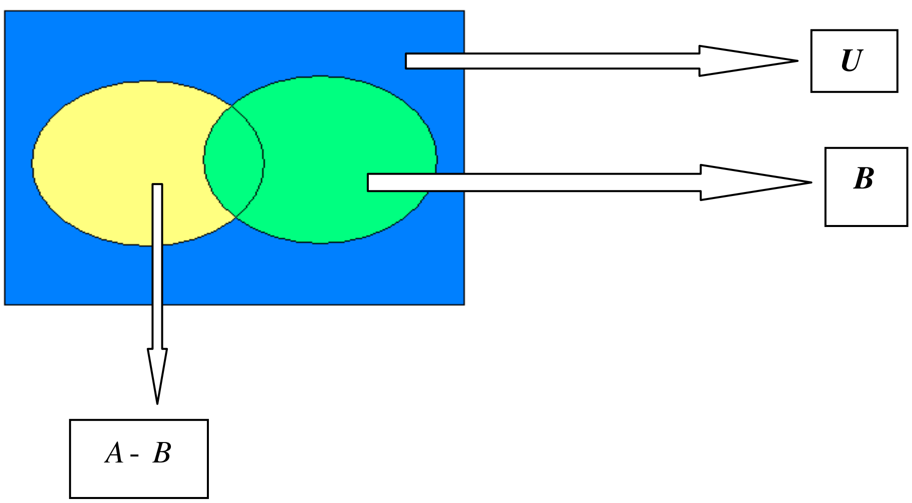
1. $\overline{\left(\overline{A}\right)} = A$ (Double negation law)
2. $A \cup \overline{A} = \overline{A} \cup A = U$ and $A \cap \overline{A} = \overline{A} \cap A = \phi$
3. If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$
4. $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (De-Morgan laws)

Therefore, we can claim that a collection of sets with respect to the set operators union, intersection and compliment forms a discrete structure.

Difference operator: Let ***A*** and ***B*** be two sets. Then difference of ***A*** and ***B*** is defined as

$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$.

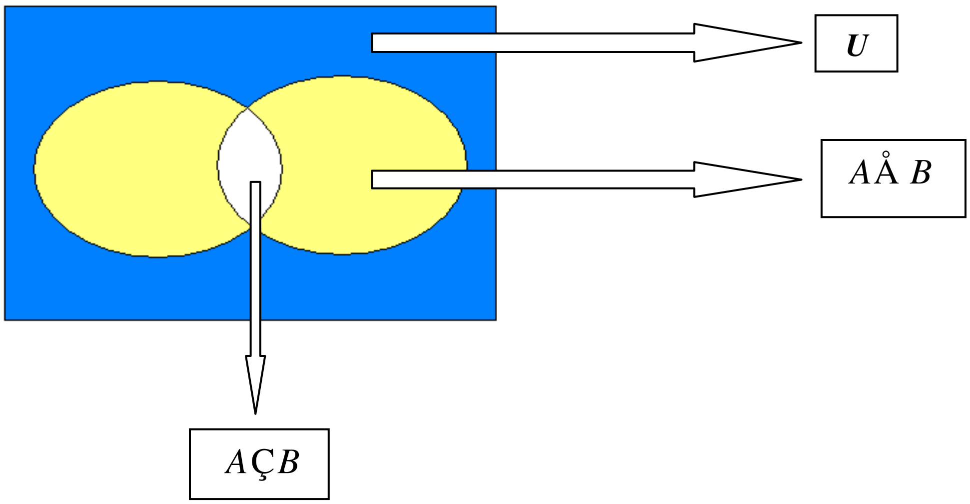
We can well claim that $A - B = \{x \in U \mid x \in A \text{ and } x \in \overline{B}\}$ so that $A - B = A \cap \overline{B}$. The Venn diagram is shown as



The membership table can be written as

A	B	$A - B$
1	**1**	**0**
1	**0**	**1**
0	**1**	**0**
0	**0**	**0**

Symmetric Difference operator: Let A and B be two sets. Symmetric Difference of A and B is defined as $A \oplus B = \{x \in U \mid x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$. Equivalently, $A \oplus B$ may be defined as $A \oplus B = (A \cup B) - (A \cap B)$ or $A \oplus B = (A - B) \cup (B - A)$. The Venn diagram can be shown as



The Membership table of $A \oplus B$ can be written as

A	B	$A \oplus B$
1	**1**	**0**
1	**0**	**1**
0	**1**	**1**
0	**0**	**0**

The following are some of the properties with respect to symmetric difference operator:

1. $A \oplus A = \phi$
2. $A \oplus B = B \oplus A$ (Commutative law)
3. $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ (Associative law)

Therefore, we can say that a collection of sets with respect to symmetric difference operators forms another discrete structure.

<u>We can have some Properties with respect to Union and Intersection Operator: These are</u>

1. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (Distributive law)
2. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$
6. $(A \cap B) \cup (C \cap D) = (A \cup C) \cap (A \cup D) \cap (B \cup C) \cap (B \cup D)$
7. $A \cup (A \cap B) = A$ (Absorption law)
8. $A \cap (A \cup B) = A$ (Absorption law)

Hence, we can claim that a collection of sets to form a discrete structure with respect to the combination of Union and Intersection Operator.

<u>Illustrative examples:</u>

1. Let A = {1, {1}, {2}}. Which of the following statements are true? Explain your answer?

$(a)\ 1 \in A,\ (b)\ \{1\} \in A,\ (c)\ \{1\} \subseteq A,\ (d)\ \{\{1\}\} \subseteq A,\ (e)\ \{2\} \in A,\ (f)\ \{2\} \subseteq A,\ (g)\ \{\{2\}\} \subseteq A$

Solution: Solution: (***a***) to (***e***) and (***g***) is true. The Statement (***f***) is false .

2 For the set A = {1, 2, 3 . . . 7}, determine the number of (i) subsets of A? (ii) proper subsets of A? (iii) Non-empty subsets of A? (iv) Non-empty proper subsets of A (v) subsets of A containing three elements? (vi) Subsets of A containing the elements1, 2 (vii) subsets of A containing 5 elements including 1 and 2?

Solution: (i) Here, set A contains totally 7 elements, the experiment consists of forming subsets using the elements of A only. Now, if we consider that B as a subset of A, then with respect to the set B, and for an element of A, there are exactly two choices; (i) an element of A under consideration is present in the set B or (ii) element not being present in B. Thus, each element of A has exactly 2 choices. Therefore, total number of subsets one can construct is $2^7 = 128$. This collection includes both the null set and the set A itself (please note this).

(ii) It is known that a proper subset means it is a set a set C such that it is a subset of A and there exists at least one element in A, not present in C. Therefore, for this reason, in the collection of subsets, we should not include the set A. Hence, number of proper subsets of A is $2^7 - 1 = 127$.

(iii) Clearly, number of non – empty subsets of A is $2^7 - 1 = 127$. This is due to the fact that we must discard the null set here.

(iv) Also, number of non – empty proper subset of A is $2^7 - 2 = 126$ since null set and the set A has to be ignored here.

(v) Now, to construct all subsets of A having exactly 3 elements. This problem is equivalent to the one, namely, in how many ways a group of 3 members may be formed from a group containing 7 persons? The answer is given by $\binom{7}{3} = \frac{7!}{3!4!} = 35$.

(vi) Here, the condition is any subset formed must include the elements 1 and 2. Therefore, choices are there for the remaining 5 elements; either to be a part of the subset or not? Hence, number of subsets containing 1 and 2 is $2^5 = 32$,

(vii) Consider a five element subset of A, say B = {- , - , - , -, -}. Now, B must include the elements 1 and 2. So, the choices of inclusion/exclusion rest with 3, 4, 5, 6, and 7 for the remaining 3 slots. These 3 slots may be filled in $\binom{5}{3} = \frac{5!}{2!3!} = 10$ ways. Thus, only 10 subsets can be formed satisfying the conditions of the problem.

3. Let S = {1, 2, 3 . . . 30}. How many subsets of A satisfy (i) |A| = 5 and the smallest element in A is 5? (ii) |A| = 5 and the smallest element in A is less than 5?

Solution: Let B = {-, -, -, -, -} be a 5 element subset of A. As the smallest element is given to be 5; the other 4 elements have to be greater than 5, these are to be selected from the remaining

numbers 6 to 30 (25 in numbers). Therefore, the answer is $\binom{25}{4} = \frac{25!}{21!4!} = 12,650.$

In the next case, the smallest element can be lower than 5 (i.e. either 1 or 2 or 3 or 4). Now if the smallest element is 1, then the other 4 numbers may be selected in C (29, 4) ways.

If the smallest element is 2, then number of ways of selecting the other 4 numbers is C (28, 4) ways. If the smallest element is 3, then we have C (27, 4) ways. When the smallest element is 4, then we have C(26, 4) ways. Thus, number of ways of doing the whole task is C (29, 4) + C (28, 4) + C (27, 4) + C (26, 4).

4. How many strictly increasing sequences of integers start with 62 and end with 92?

Solution: Consider a sequence of numbers, say $a_1 < a_2 < a_3 < \ldots\ldots\ldots\ldots < a_n$ with $a_1 = 62$ and $a_n = 92$. The possible sequence of numbers are the following: (62, 92), (62, 63, 92), (62, 78, 89, 92), (62, 63, 64, 65,. . 89, 90, 91, 92). In the last one all the numbers in between 62 and 92 are included. From these, it is clear that during the construction process, we have included some numbers in the sequence and at the same time, we have ignored the numbers between 62 and 92. Clearly there are 27 numbers in between 62 and 92, therefore answer to this question is $\mathbf{2^{27}}$.

5. For U = {1, 2, 3 . . . 9}, A = {1, 2, 3, 4, 5}, B = {1, 2, 4, 8}, C = {1, 2, 3, 5, 7} and D= {2, 4, 6, 8}, compute the following: (a) $(A \cup B) \cap C$, (b) $A \cup (B \cap C)$, (c) $\overline{C} \cup \overline{D}$ (d) $\overline{C \cup D}$, (e) $(A \cup B) - C$, (f) $B - (C - D)$.

Solution: Solution: (a) Note that $A \cup B = \{1, 2, 3, 4, 5, 8\}$ so that $(A \cup B) \cap C = \{1, 2, 3, 5\}$

.

Observe that $B \cap C = \{1, 2\}$ thus, $A \cup (B \cap C) = A = \{1, 2, 3, 4, 5\}$. (d) By De-Morgan law, $\overline{C} = \{4, 6, 8, 9\}$, $\overline{D} = \{1, 3, 5, 7, 9\}$, hence $\overline{C} \cup \overline{D} = \{1, 3, 4, 5, 6, 7, 8, 9\}$. (d) By De – Morgan laws, $\overline{C \cup D} = \overline{C} \cap \overline{D} = \{9\}$. (e) With $A \cup B = \{1, 2, 3, 4, 5, 8\}$ and C = {1, 2, 3, 5, 7}, yields $(A \cup B) - C = \{4, 8\}$. (f) It is given that B = {1, 2, 4, 8} and C ={1, 2, 3, 5, 7},
D = {2, 4, 6, 8} gives C – D ={1, 3, 5, 7} so that B – (C -D) = {2, 4, 8}.

6. Determine the sets A and B, given that $(A \cup B = \{1, 2, 4, 5, 7, 8, 9\}$, and $A - B = \{1, 2, 4\}$, $B - A = \{7, 8\}$

Solution: Clearly, A = {1, 2, 4, 5, 9} and B = {5, 7, 8, 9}.

7. Determine the sets A, B and $A \cup B$ **given that** $A \cap B = \{4, 9\}$, $A - B = \{1,\ 3,\ 7,\ 11\}$, $B - A = \{2,\ 6,\ 8\}$

Solution: $A = \{1, 3, 4, 7, 9, 11\}$ $B = \{2, 4, 6, 8, 9\}$ $A \cup B = \{1, 2, 3, 4, 6, 7, 8, 9, 11\}$

8. Prove or disprove the following: For the sets $A, B, C \subseteq U$, $A \cup C = B \cup C$ **implies** $A = B$? **for sets** $A, B, C \subseteq U$, $A \cap C = B \cap C$ **implies** $A = B$?

Solution: **Consider** $U = \{a, b, c, d, e, f, g, h\}$, $A = \{d, g\}$, $B = \{a, e, g\}$, $C = \{a, d, e\}$, observe that $A \cup C = \{a, d, e, g\}$ and $B \cup C = \{a, d, e, g\}$ but clearly A and B are different. It may be recalled that this property holds for a set of numbers with respect to operation usual addition, namely, $a + c = b + c$, then $a = b$. Why not here? any explanation? For the next problem, set up $U = \{a, b, c, d, e, f, g, h\}$, $A = \{d, g\}$, $B = \{a, e, g\}$, $C = \{a, d, e\}$, note that $A \cap C = B \cap C = \{a, e\}$ and as usual A and B are different sets. Can you explain me why these properties are not working with sets and with respect to union or intersection operator? A similar property is true with respect to usual multiplication operator, namely, $a \cdot c = b \cdot c$ implying that $a = b$, if $c \neq 0$.

9. Using membership table, verify whether $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Solution: We shall set up $x : 1$ means $x \in A, B$ and $x : 0$ means $x \notin A, B$. Consider the membership table of $\overline{A \cup B} = \overline{A} \cap \overline{B}$,

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	**0**	0	0	**0**
1	0	1	**0**	0	1	**0**
0	1	1	**0**	1	0	**0**
0	0	0	**1**	1	1	**1**

From the above membership table (comparison of 4^{th} column and the last column, it is clear that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

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